Lecture 9: Intro to ∞-categories.

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Contents

1 Motivation.

$\mathcal{A}$ abelian category (Mod_R). We can form the derived category $D(\mathcal{A})$ which is source of cohomological invariants. We also the category of spectra Sp. These are both triangulated categories.
Issues with triangulated categories: If T, T' are triangulated categories, then $\mathrm{Fun}(T, T')$ in general is not triangulated.
Also there is no good theory of ring objects or algebra objects in a triangulated category.
Also given $\mathcal{A} \to T$ a exact functor i.e sends SES to exact triangles, then it does not extend to $D(\mathcal{A}) \to T$.

To solve these, it is good to find an ∞-enhancement of Δ-categories i.e there exist an "∞"-category such that $\mathrm{Ho}(\mathcal{C}) \cong T$.
Turns out that $\mathrm{Sp}, D(\mathcal{A})$ have (unique) ∞-enhancements.
Turns out that ∞-categories will have better functorial behaviour and easy to work with in terms of universal properties.

2 What they should be?

2-categories. An example of that will be Cat.

An n-category is a category with additional data of $i+1$-morphisms.
We can say that 2-categories is a category which is enriched in Cat.

In general, we do not mind about equality of functions but we are ok with natural transformation.
But, we need to have an extra data of such isomorphisms.
$X \in \text{Top}$, $\pi_{\leq n}(X)$ has :

1. Objects : $x \in X$ points.
2. 1- Morphisms are just paths.
3. 2-Morphisms are homotopies of homotopies.
4. n- morphisms homotopy classes of $n-1$ homotopies.

3 What are models of ∞-categories?

Models:

1. Complete segal spaces.
2. $\text{Cat}_{Top} = \text{Cat}_{sSet}$.
3. Model categories.
4. Quasi-categories.

All these models are equivalent in some sense. We have the following functors :

$$\text{Cat} \hookrightarrow \text{sSet} \hookleftarrow \text{Top}$$

given by N and $\text{Sing}(-)$ respectively.
We have Δ^n standard n-simplex.
and we define $\text{Sing}_n(X) = \text{Maps}_{Top}(|\Delta^n|, X)$.

Kan condition: $K \in \text{sSet}$ satisfies the Kan condition if for all $0 \leq i \leq m$, we have the dotted arrow:

$$\begin{array}{ccc} \Lambda_i^m & \longrightarrow & K \\ \big\downarrow & \overset{\exists}{\nearrow} & \\ \Delta^m & & \end{array}$$

where Λ_i^m : union of all faces of Δ^m except the one containing i.
Facts :$\text{Sing}_\bullet(X)$ for $X \in \text{Top}$. $\text{N}(C)$ does not satisfy the Kan condition.

We can see that if $\mathcal{C}$ is a groupoid, then the Kan condition holds. So it makes sense to have the following weaker condition, i.e the Kan condition for $0 < i < m$.
Thus, we arrive at the definition:

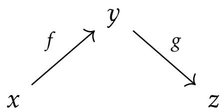
Definition 3.0.1. An ∞-category is a simplicial set K satisfying the weaker Kan condition.

Example 3.0.1. $N(\mathcal{C})$ is an ∞-category.

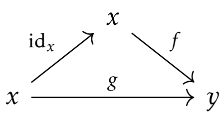
4 Basic terminologies.

Definition 4.0.1. $\mathcal{C}$ be an ∞-category. The Objects are : $\Delta^0 \to \mathcal{C}$ i.e $\mathcal{C}_0$.
Morphisms are $\Delta^1 \to \mathcal{C}$.

Definition 4.0.2. $f : x \to y; g : y \to z$, then the composition is the filling of the inner horn $\Lambda^2_1 \to \mathcal{C}$ to $\Delta^2 \to \mathcal{C}$ where the morphism $\Lambda^2_1 \to \mathcal{C}$ is given by



Definition 4.0.3. $f, g : x \to y$ are two morphisms then we say f is homotopic to g , if there exists a 2-simplex of the form :



We show that any two compositions are homotopic to each other.

Thus, we can define the **homotopy category** of an ∞-category as :

1. Ob(Ho($\mathcal{C}$)) = Objects of $\mathcal{C}$.
2. Morphisms are homotopy classes of morphisms of $\mathcal{C}$.

Definition 4.0.4. $\mathcal{C}, \mathcal{D}$ ∞-categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is an arbitrary map of simplicial sets.

5 Joins.

At first, let us recall joins in classical category theory.

Definition 5.0.1. Let $\mathcal{C}, \mathcal{C}'$ be categories. We define the join of $\mathcal{C}$ and $\mathcal{C}'$, denoted by $\mathcal{C} * \mathcal{C}'$ as follows:

1. Objects: ob($\mathcal{C}$) or ob($\mathcal{C}'$).
2. $$\mathrm{Hom}_{\mathcal{C} * \mathcal{C}'}(X, Y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X, Y) & X, Y \in \mathcal{C} \\ \mathrm{Hom}_{\mathcal{C}'}(X, Y) & X, Y \in \mathcal{C}' \\ \phi & X \in \mathcal{C}', Y \in \mathcal{C} \\ * & X \in \mathcal{C}, Y \in \mathcal{C}' \end{cases}$$

We now define joins for simplicial sets which will be joins for ∞-categories.

Definition 5.0.2. Let S, S' be simplicial sets. We define the **join** of S, S', denoted by $S * S'$ as

$$(S * S')_n = S_n \cup S'_n \cup \bigcup_{i+j=n-1} S_i \times S'_j$$

Example 5.0.1. We have $\Delta^j * \Delta^i \cong \Delta^{i+j-1}$.

We have the following proposition.

Proposition 5.0.1. *Let S, S' be ∞-categories. Then the join $S * S'$ is also an ∞-category.*

Proof. Let $p : \Lambda^n_i \to S * S'$ be the map. If p maps entirely into S or S', then we have the extension by the property of S, S' being ∞-categories.
Now suppose p maps vertices $0, 1, ..j$ to S and $j+1, j+2, ...n$ to S', by extension property of S and S' we have the maps :

$$p_j : \Delta^{\{0,1,2..j\}} \to S, p'_j : \Delta^{\{j+1,j+2,..n\}} \to S'$$

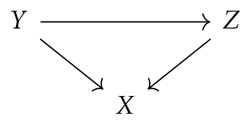
Together we get a map $\Delta^n \to S * S'$. □

We shall use the following useful notation:

Notation 5.0.1. Let K be a simplicial set. Then $K^{\triangleleft}$ is called the **left cone** defined by $\Delta^0 * K$. Dually we call $K^{\triangleright}$ to be the **right cone** which is defined by $K * \Delta^0$.

6 Over and undercategories of an ∞-category.

In classical category thery, we have the concept of over and undercategories. Given a category $\mathcal{C}$ and an object $x \in \mathcal{C}$, we have the overcategory defined as :

1. Objects :$Y \to X$ morphisms in $\mathcal{C}$.
2. Morphisms : 

Choosing an object X is a map of simplicial sets $X : \Delta^0 \to N(\mathcal{C})$. The analogue of objects and morphisms of overcategory for $N(\mathcal{C})$ can be defined as follows:

1. Objects $p_0 : \Delta^0 * \Delta^0 \to N(\mathcal{C})$ where the restriction on second factor gives X.
2. Morphisms $p_1 : \Delta^1 * \Delta^0 \to N(\mathcal{C})$ where restricting on second factor gives X.

This motivates us to give the following definition of overcategory for simplicial sets and thus ∞-categories.

Definition 6.0.1. Given $p : K \to S$ an aribtrary map. There exists a simplicial set $S_{/p}$ which is defined as follows:

$$(S_{/p})_n = \mathrm{Hom}_p(\Delta^n * K, S)$$

where the subscript on the right hand side indicates that we consider only morphisms $f : \Delta^n * K \to S$ such that $f|_K = p$. $S_{/p}$ is called the **overcategory** associated to p.

Dually, we can define the undercategory with respect to a map of simplicial sets.

Definition 6.0.2. We define $S_{p/}$, the **undercategory** associated to p by replacing $\Delta^n * K$ with $K * \Delta^n$ in the previous definition.

Remark 6.0.1. Let $\mathcal{C}$ be an ordinary category. Let X be an object of $\mathcal{C}$ (also an object of $N(\mathcal{C})$). We have the following equivalence $N(\mathcal{C})_{/X} \cong N(\mathcal{C}_{/X})$.

7 Initial and Final Objects of an ∞-category.

Let X be an object in an ordinary category $\mathcal{C}$. Then X is said to be final if for all objects $Y \in \mathcal{C}$, we have an unique morphism $f_Y : Y \to X$. X is said to be initial if it is final in the category $\mathcal{C}^{op}$.

In the language of ∞-categories, we define such notions in the following manner.

Definition 7.0.1. Let $\mathcal{C}$ be an ∞-category, a vertex $x \in \mathcal{C}$ (i.e $x \in \mathcal{C}_0$) is said to be **final**/ **initial** if for any map $f : \partial\Delta^n \to \mathcal{C}$ such that $f([n]) = x(f([0]) = x)$, then it extends to a morphism $\tilde{f} : \Delta^n \to \mathcal{C}$.

Remark 7.0.1. In the case of $n = 1$, we see the condition in the case of classical category theory. In fact, it can be see that the for an object $y \in \mathcal{C}$ and if x is initial/final object of the category, the mapping space $\mathrm{Hom}_R(x, y)$ ($\mathrm{Hom}_R(y, x)$) is contractible.

8 Diagrams, Limits and colimits in an ∞-category.

In classical category theory, a diagram is a morphism from an indexing category to the category. In the simplicial world, it is defined as follows:

Definition 8.0.1. A diagram from a simplicial set K to a simplicial set C is a morphism of simplicial sets $F : K \to C$.

A limit of a diagram in the context of classical category theory is defined via the universal property. An elegant way to view this is to see in the following manner:
Let $\mathcal{C}$ be a category and I be an indexing category. Let $F : I \to \mathcal{C}$ be a functor Consider the overcategory $\mathcal{C}$ over F denoted as $\mathcal{C}_F$. This is defined as :

1. Objects : {Diagrams : $\gamma_{C,i} : F(C) \to F(I)\}_{C \in \mathcal{C}}$ with usual commutativity relations.
2. Morphisms : Morphisms between diagrams with commutativity conditions.

Note that this is the special cases of overcategories over a morphism in the case of simplicial sets.
Now a limit of the functor F is the final object in the category $\mathcal{C}_F$. The usual universal property translates into the condition of the object being final in this category.

We use this analogy to extend the notion in the ∞-categorical setting.

Definition 8.0.2. Let $\mathcal{C}$ be an ∞-category. Let $f : K \to \mathcal{C}$ be an arbitrary map of simplicial sets. Then a **limit** of f is a final object in the overcategory $\mathcal{C}_{/f}$.
Dually, a **colimit** of F is an initial object in the undercategory $\mathcal{C}_{f/}$.

We may identify the colimit as an object as a map $\bar{p} : K^{\triangleright} \to \mathcal{C}$ extending p. In general, we will say a map $\bar{p} : K^{\triangleright} \to \mathcal{C}$ is a **colimit diagram** if it is a colimit of $p = \bar{p}|_K$. Dually, we have notion of a limit diagram.

Remark 8.0.1. Let us spell out the definitions in a special case. Let I be the three object category considered as a simplicial set:

$$\begin{array}{ccc} & & i_1 \\ & & \downarrow \\ i_2 & \longrightarrow & i_0 \end{array}$$

Let $F : I \to \mathcal{C}$ be a functor where $\mathcal{C}$ is an ∞-category. Denote the diagram as :

$$\begin{array}{ccc} & & X_1 \\ & & \big\downarrow{\scriptstyle f_{10}} \\ X_2 & \xrightarrow{f_{02}} & X_0 \end{array}$$

We want to see what an universal property of the limit of F is. Consider the overcategory $\mathcal{C}_{/F}$. Let X be the limit of F. Then explicitly writing the definition of the limit, we get the following diagram:

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{f_n} & \mathcal{C}_{/F} \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

where f_n is a map which sends $[n] \to X$. Let us spell it out case by case what this means:

1. **n=0**: In this case, just says that we have a unique morphism $\Delta^0 \to \mathcal{C}_{/F}$, i.e a map $\Delta^0 * I \to \mathcal{C}$ such that when restricted to I it is F. Thus we have a following diagram :

$$\begin{array}{ccc} X & \xrightarrow{f_1} & X_1 \\ \big\downarrow{\scriptstyle f_2} & \searrow^{f_0} & \big\downarrow{\scriptstyle f_{10}} \\ X_2 & \xrightarrow{f_{20}} & X_1 \end{array}$$

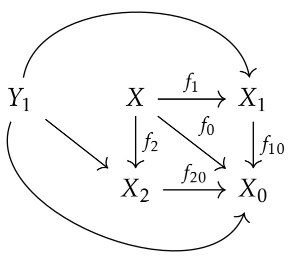
along with homotopies on the triangles which are 2-simplexes in the ambient category $\mathcal{C}$.

2. **n=1**: In this case, we are given a map $\partial\Delta^1 \to \mathcal{C}_{/F}$ which sends $[1] \to X$, thus we have a morphism of simplicial sets $\partial\Delta^1 * I \to \mathcal{C}$ such that :

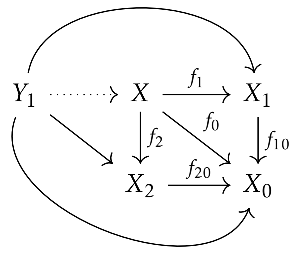
   (a) $[1]$ should be sent to X.

   (b) When restricted to I, it should be F.

   Unravelling this morphism as a diagram, we get the following:



   with additional homotopies on the triangles $Y_1X_2X_0$ and $Y_1X_1X_0$.
   The existence of the morphism $\Delta^1 \to \mathcal{C}_{/F}$ now translates to existence of the dotted arrow in the diagram :



such that we have 2-simplices: Y_1XX_1, YXX_2 and YXX_0 and 3-simplices : YXX_1X_0 and YXX_2X_0.

Notice this is exactly a homotopic generalization of the universal property of limits in classical category theory.

3. For cases $n = 2$ and higher cases, we need to replace Y_1 and X_1 by an boundary of an n-simplex where the nth coordinate is X. The existence of the unique morphism says that we can fill the boundary to get an n-simplex and additional homotopies in the diagram which are compatible with one another.