Motivic homotopy theory of algebraic stacks

Dissertation

zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften (Dr.rer.nat.)

von

Chirantan Chowdhury, M.Sc.

geboren in Porbandar, Indien

vorgelegt beim Fachbereich Mathematik
Universität Duisburg-Essen
Campus Essen

Gutachter

Prof. Dr. Jochen Heinloth
Prof. Dr. Markus Spitzweck

Essen 2021

Date of Defense: 18$^{\text{th}}$ August, 2021

Abstract

In this thesis, we extend the definition of motivic homotopy theory from schemes to a large class of algebraic stacks and establish a six functor formalism. The class of algebraic stacks that we consider includes many interesting examples: quasi-separated algebraic spaces, local quotient stacks and moduli stacks of vector bundles. We use the language of ∞-categories developed by Lurie to extend the definition of motivic homotopy theory. Morever, we use the so-called 'enhanced operation map' due to Liu and Zheng to extend the six functor formalism from schemes to our class of algebraic stacks. We also prove that six functors satisfy properties like homotopy invariance, localization and purity.

Zusammenfassung

In dieser Arbeit erweitern wir die Definition motivischer Homotopietheorie von Schemata auf eine große Klasse algebraischer Stacks und etablieren einen Sechs-Funktor-Formalismus. Die Klasse algebraischer Stacks, die wir betrachten, enthält viele interessante Beispiele: quasi-separierte algebraische Räume, lokale Quotientenstacks und Modulstacks von Vektorbündeln. Wir verwenden die Sprache der ∞-Kategorien, entwickelt von Lurie, um die Definition von motivischer Homotopietheorie zu erweitern. Ferner benutzen wir die sogenannte 'enhanced operation map' von Liu und Zheng, um den Sechs-Funktor-Formalismus von Schemata auf unsere Klasse algebraischer Stacks auszuweiten. Wir zeigen außerdem, dass die sechs Funktoren Eigenschaften wie beispielsweise Homotopieinvarianz, Lokalisierung und Reinheit erfüllen.

CONTENTS

CHAPTER 1

INTRODUCTION

The six functor formalism was formulated by Grothendieck to give a framework for the basic operations and duality statements for cohomology theories. In brief, a six functor formalism is a theory of coefficient systems relative to any scheme with a collection of six functors $f^*, f_*, f^!, f_!, \otimes, \mathrm{Hom}$ which satisfy a set of relations. This formalism is usually formulated in the language of triangulated categories. In [MV99], Morel and Voevodsky define the general theory of $\mathbb{A}^1$-homotopy theory of schemes which incorporates homotopy theoryin the field of algebraic geometry. To a scheme S, they associate a triangulated category $\mathrm{SH}(S)$ which is defined by applying $\mathbb{A}^1$-localization and $\mathbb{P}^1$-stabilization to the category of simplicial Nisnevich sheaves. Voevodsky and Ayoub ([Ayo07a] and [Ayo07b]) constructed a six functor formalism of $\mathbb{A}^1$-homotopy theory. In this thesis, we extend the definition of $\mathcal{SH}$ to a large class of algebriac stacks and provide a six functor formalism for $\mathcal{SH}$ using the language of ∞-categories developed by Lurie ([Lur09] and [Lur17]).

In order to motivate the need of language of ∞-categories, let us recall the six functor formalism of derived categories of ℓ-adic sheaves over an algebraic stack. To an algebraic stack $\mathcal{X}$, one can define the derived category of the algebraic stack $\mathcal{X}$ as derived category of ℓ-adic étale sheaves over $\mathcal{X}$. For example, if $\mathcal{X} = \mathrm{B}\mathbb{G}_m$, then the derived category of $\mathrm{B}\mathbb{G}_m$ is the derived category of $\mathbb{G}_m$-equivariant ℓ-adic étale sheaves over a point. As the connected group $\mathbb{G}_m$ cannot act non-trivially on locally constant sheaves, this is equivalent to the category of sheaves over a point. Thus this naive definition implies that $\mathrm{D}(\mathrm{B}\mathcal{G}_m)$ is equivalent to the derived category of a point. But we have

$$H^*(\mathrm{B}\mathbb{G}_m) \cong \mathbf{Q}_l[c]$$

where c is in degree 1 ([Tot99]).

In [LO08b] and [LO08a], Laszlo and Olsson define derived categories of algebraic stacks and construct the six functor formalism using the lisse-étale topos. They use simplicial methods to to construct the derived category that gives the expected answer for the cohomology of $\mathrm{B}\mathbb{G}_m$. The fact that the lisse-étale topos is not functorial makes the construction of derived pullback bit technical.The language of ∞-categories allows us to circumvent this problem.

In [LZ17], Liu and Zheng construct a six functor formalism of derived ∞-categories of ℓ-adic sheaves for any algebraic stack. To any scheme X, the derived ∞-category $\mathcal{D}_{et}(X, \mathbf{Q}_l)$ is the ∞-categorical enhancement of the usual derived category. The major advantange of the ∞-categorical language is that the derived ∞-category satisfies étale descent. For any algebraic stack $\mathcal{X}$, the ∞-category $\mathcal{D}_{et}(\mathcal{X}, \mathbf{Q}_l)$ consturcted by Liu and Zheng is isomorphic to the limit of derived ∞-categories over Čech nerve of any atlas $x : X \to \mathcal{X}$. In other words, we have

$$\mathcal{D}_{et}(\mathcal{X}, \mathbf{Q}_l) \cong \lim \Big(\mathcal{D}_{et}(X, \mathbf{Q}_l) \rightrightarrows \mathcal{D}_{et}(X \times_{\mathcal{X}} X, \mathbf{Q}_l) \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \mathcal{D}_{et}(X \times_{\mathcal{X}} X \times_{\mathcal{X}} X, \mathbf{Q}_l) \qquad \cdots \Big) \tag{1.1}$$

where the maps in the limit are the derived pullback maps. Their construction uses abstract descent theory of the language of ∞-categories. This also allows to construct the pullback functor in a canonical way. Morever, they prove that their formalism agrees with the one introduced by Laszlo and Olsson once one passes to homotopy categories of the derived ∞-categories. Thus the language of ∞-categories seem advantageous to extend ∞-sheaves from schemes to algebraic stacks. We shall use a similar technique in our setting of motivic homotopy theory but in this case extra care is needed because motivic invariants usually do not satisfy étale descent.

To a Noetherian scheme of finite Krull dimension S, the motivic stable homotopy category $\mathcal{SH}^{\otimes}(S)$ is a presentable stable symmetric monoidal ∞-category (we refer to [Rob15] for the notations). The functorial assignment makes $\mathcal{SH}^{\otimes}$ into a functor

$$\mathcal{SH}^{\otimes} : \mathrm{N}(\mathrm{Sch}_{fd})^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{stb}) \tag{1.2}$$

where the target is the ∞-category of stable presentable symmetric monoidal ∞-categories. As mentioned above, we cannot use equation Eq. (1.1) as a definition of $\mathcal{SH}^{\otimes}$ for an algebraic stack because $\mathcal{SH}^{\otimes}$ does not satisfy étale descent and thus Eq. (1.1) would depend on the choice of the atlas X. We resolve this problem by specifying a class of smooth atlases for which we can prove descent. The resulting class of $(2,1)$-category of algebraic stacks Nis-locSt consists of algebraic stacks which admit an atlas admitting Nisnevich-local sections. This includes all quasi-compact, quasi-separated algebraic spaces, quotient stacks $[X/G]$ where G is an affine algebraic group, local quotient stacks, the moduli stack of vector bundles Bun_n, the moduli stack of G-bundles Bun_G and moduli space of stable maps. Using the formulation of enhanced operation map ([LZ17]), we also manage to extend the six functor formalism from schemes to Nis-locSt. Our main result is as follows (see Theorem 5.5.1 for a complete statement):

Theorem 1.0.1. *The functor $\mathcal{SH}^{\otimes}(-)$ extends to a functor*

$$\mathcal{SH}^{\otimes}_{ext} : \mathrm{N}^{D}_{\bullet}(\text{Nis-locSt})^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{stb}).$$

Morever,

1. *For any* $\mathcal{X} \in \text{Nis-locSt}$, *there exist functors* $\otimes, \text{Hom} : \mathcal{SH}_{\text{ext}}(\mathcal{X}) \times \mathcal{SH}_{\text{ext}}(\mathcal{X}) \to \mathcal{SH}_{\text{ext}}(\mathcal{X})$.

2. *For any morphism* $\mathsf{f} : \mathcal{X} \to \mathcal{Y}$ *in* Nis-locSt, *there is a pair of adjoint functors*

$$\mathsf{f}^* : \mathcal{SH}_{\text{ext}}(\mathcal{Y}) \to \mathcal{SH}_{\text{ext}}(\mathcal{X}) \ , \ \mathsf{f}_* : \mathcal{SH}_{\text{ext}}(\mathcal{X}) \to \mathcal{SH}_{\text{ext}}(\mathcal{Y}).$$

3. *For a morphism* $\mathsf{f} : \mathcal{X} \to \mathcal{Y}$ *in* Nis-locSt *which is separated of finite type and representable by algebraic spaces, there is a pair of adjoint functors*

$$\mathsf{f}_! : \mathcal{SH}_{\text{ext}}(\mathcal{X}) \to \mathcal{SH}_{\text{ext}}(\mathcal{Y}) \ , \ \mathsf{f}^! : \mathcal{SH}_{\text{ext}}(\mathcal{Y}) \to \mathcal{SH}_{\text{ext}}(\mathcal{X}).$$

These functors restrict to the known functors on the category of schemes. Furthermore, the projection formula, base change, localization, homotopy invariance and purity extend to Nis-locSt.

Hoyois defines SH for quotient stacks of the form $[\mathsf{X}/\mathsf{G}]$ where G is tame ([Hoy17]). His construction apriori depends on the presentation of the stack. Our construction allows us to drop the tameness assumption for quotient stacks and provides a version of $\mathcal{SH}$ that does not depend on the choice of a presentation.

We now give a brief outline of the chapters in the thesis.

1. In Chapter 2, we prove an ∞-categorical generalization of the statement we have descent along morphisms admitting sections. We also provide a skeletal description of the split-simplicial category $\Delta_{-\infty}$. The key result follows from the statement that split-simplicial objects are colimit diagrams ([Lur09, Lemma 6.1.3.16]).

2. In Chapter 3, we enhance the motivic stable homotopy functor from schemes to algebraic stacks. This follows from Theorem 3.4.1 which is set in an abstract setup of categories of stacks admitting $\mathcal{T}$-local sections (Definition 3.2.1). The key example of categories of stacks admitting $\mathcal{T}$-local sections is the $(2,1)$-category Nis-locSt. It is important to note that Theorem 3.4.1 is a special case of [LZ17, Proposition 4.1.1]. We give a new proof of the theorem which is partly inspired from the proof of Liu and Zheng. The extension theorem also allows to construct the four functors $\mathsf{f}_*, \mathsf{f}^*, \text{Hom}$ and $\otimes$.

3. In Chapter 4, we construct the exceptional functors $\mathsf{f}_!$, $\mathsf{f}^!$ and also prove base change and projection formulas. This is proved by the so called enhanced operation map introduced by Liu and Zheng in [LZ17]. In short, we extend the enhanced operation from schemes to Nis-locSt in the setting of categories of stacks admitting $\mathcal{T}$-local sections (Theorem 4.1.1). The extension is a special case of the DESCENT program ([LZ17, Chapter 4]) and we give a new proof of the extension of the enhanced operation from schemes Nis-locSt which gives us the exceptional functors, projection formula and base change.

4. In Chapter 5, we prove the other relations of the six functors namely smooth and proper base change, localization, homotopy invariance and purity. We then collect all the results in a single theorem (Theorem 5.5.1).

5. In Appendix A, we recall the notions of ∞-categories that we need in the thesis. In particular, we recall the notion of Kan extensions, presentable ∞-categories, ∞-sheaves and stable ∞-categories ([Lur09],[Lur17] and [Lur18b]).

6. In Appendix B, we recall the notion of symmetric monoidal ∞-categories in terms of ∞-operads ([Lur17, Chapter 2]) and module objects over commutative algebra objects associated to an operad $\mathcal{C}^{\otimes}$.

7. In Appendix C, we briefly recall the notions of motivic homotopy theory of schemes in the language of ∞-categories ([Rob15], [Rob15] and [Rob14]). We also recall the six operations and give a brief sketch of the construction of purity transformation $\mathsf{\rho_f}$ and the pushforward transformation $\mathsf{\alpha_f}$.

8. In Appendix D, we explain the construction of the enhanced operation map due to Liu and Zheng ([LZ17]) which we use in Chapter 3. At first, we review the ∞-categorical generalization of Deligne's compactification ([LZ12, Theorem 0.1]). We also give a brief idea of how the proof of the theorem. Then we recall the notion of partial adjoints ([LZ17, Proposition 1.4.4]). With the compactification theorem and partial adjoints, we end the chapter with the construction of enhanced operation map.

Acknowledgements.

I cannot begin to express my thanks to my advisor Jochen Heinloth for his invaluable advice, continuous support and patience during my PhD study. His insightful feedback pushed me to sharpen my thinking and brought the thesis to its present shape.
I would like to extend my sincere thanks to Markus Spitzweck, who agreed to be my referee. I would also like to thank Marc Levine for helpful discussions regarding my thesis. I also thank Jonas Franzel for checking latex and grammatical errors in my thesis.
I would like to extend my gratitude to all the members of ESAGA group for providing an amazing atmosphere and giving me the opportunity to learn a variety of topics via Forschüngseminars, Oberseminars and PhD seminars. A special thanks to Alessandro, Andrea A., Andrea M., Antonio, Aprameyo, Daniel, Felix, Gabriela, Gürkan, Konstantin, Lennart, Luca D., Luca M., Louis, Matteo, Martin, Nicolas, Nils E., Paulina, Pietro, Ran, Robin, Tommaso and Xucheng.
I must also thank Anandam, Antara, Nirabhra and Lucia who provided encouragement and motivation along the journey of my PhD.
Finally, I would like to thank my parents for their unparalleled love and support. I am forever indebted to them for giving me opportunitues and experiences that have made me who I am.

CHAPTER 2

DESCENT ALONG SECTIONS

In this chapter, we recall the infinity categorical setup generalizing the classical statement that descent along morphisms that admit sections is usually automatic. We begin with the corresponding statements for usual categories and then explain the higher categorical analog. We include a skeletal description of the split-simplicial category $\Delta_{-\infty}$ which we could not find in the literature. The main result is then due to Lurie ([Lur09, Lemma 6.1.3.16]).

2.1 Split forks.

Definition 2.1.1. 1. A *fork* in a category $\mathcal{C}$ is a diagram of the form

$$a \xrightarrow{d^0_{-1}} b \underset{d^1_1}{\overset{d^1_0}{\rightrightarrows}} c \tag{2.1}$$

where $d^1_0 \circ d^0_{-1} = d^1_1 \circ d^0_{-1}$.

2. A fork is an *equalizer* if a is the limit of the diagram

$$b \underset{d^1_1}{\overset{d^1_0}{\rightrightarrows}} c. \tag{2.2}$$

There are dual notions of forks and equalizers which are called *coforks* and *coequalizers*.

Example 2.1.2. 1. Let $F : \mathcal{C}^{op} \to$ Sets be a presheaf on a site $\mathcal{C}$ admitting products. Then F is a sheaf if for a covering map $U \to X$, the diagram

$$F(X) \longrightarrow F(U) \rightrightarrows F(U \times_X U) \tag{2.3}$$

is an equalizer.

2. Let $f : A \to B$ be a morphism of commutative rings. The diagram

$$A \xrightarrow{f} B \underset{\{1\otimes b\}}{\overset{\{b\otimes 1\}}{\rightrightarrows}} B \otimes_A B \tag{2.4}$$

is a fork.

We now move to the definition of a split fork.

Definition 2.1.3. A fork

$$a \xrightarrow{d^0_{-1}} b \underset{d^1_1}{\overset{d^1_0}{\rightrightarrows}} c \tag{2.5}$$

is a *split fork* if it can be embedded into a diagram

$$a \underset{s^{-1}_{-1}}{\overset{d^0_{-1}}{\rightleftarrows}} b \underset{d^1_1}{\overset{d^1_0}{\rightrightarrows}} c, \qquad s^0_{-1} : c \to b \tag{2.6}$$

where

1. $s^{-1}_{-1} d^0_{-1} = \mathrm{id}_a$,
2. $s^0_{-1} d^1_0 = \mathrm{id}_b$,
3. $s^0_{-1} d^1_1 = d^0_{-1} s^{-1}_{-1}$.

Lemma 2.1.4. *A split fork is an equalizer.*

Proof. Let

$$a \underset{s^{-1}_{-1}}{\overset{d^0_{-1}}{\rightleftarrows}} b \underset{d^1_1}{\overset{d^1_0}{\rightrightarrows}} c, \qquad s^0_{-1} : c \to b \tag{2.7}$$

be a split fork. We need to show it is a equalizer i.e. a is the limit of the diagram

$$b \underset{d^1_1}{\overset{d^1_0}{\rightrightarrows}} c. \tag{2.8}$$

Let $h : x \to b$ be a morphism such that $d^1_0 h = d^1_1 h$. If we denote $s^{-1}_{-1} h : x \to a$ by h'', then h factors through h'' because

$$d^0_{-1} h'' = d^0_{-1} s^{-1}_{-1} h = s^0_{-1} d^1_1 h = s^0_{-1} d^1_0 h = h. \tag{2.9}$$

The lifting of the map h is unique as $s^{-1}_{-1} d^0_{-1} = \mathrm{id}_a$. $\square$

Example 2.1.5. Let $F : \mathcal{C}^{op} \to \text{Sets}$ be a presheaf where $\mathcal{C}$ is a site for which the coverings are morphisms $U \to X$ which admit a section. Then F is automatically a sheaf because

$$F(X) \longrightarrow F(U) \rightrightarrows F(U \times_X U) \tag{2.10}$$

is a split fork. The splitting maps are induced by the section of the map $U \to X$.

Remark 2.1.6. Split forks play an important role in Barr–Beck Monadicity theorem ([Lur17, Theorem 4.7.3.5]). Split forks arise in the proof of faithfully flat descent.

2.2 Split-simplicial objects.

In this section, we give the simplicial analogs of forks and split forks: *augmented* and *split-simplicial objects*. Augmented simplicial objects have the potential to be colimit diagrams. As split-simplicial objects are simplicial analogs of split forks (which are limit diagrams by Lemma 2.1.4), it is natural to expect them to be colimit diagrams. We shall recall the definitions of the categories Δ_+ and $\Delta_{-\infty}$ and give a skeletal description of all of these categories.

Definition 2.2.1. [Lur09, Definition 6.1.2.2] The *augmented simplicial category* Δ_+ is defined as follows:

1. Objects: $\{[n]_+ = [n] \cup \{-\infty\}; n \geq -1\}$ where $-\infty$ is the minimal element in $[n]_+$, in particular $[-1]_+ = \{-\infty\}$.

2. Morphisms: $\text{Hom}_{\Delta_+}([m]_+, [n]_+) = \{\alpha : [m]_+ \to [n]_+ \,; \alpha \ \text{ non-decreasing} \,;\ \alpha^{-1}(-\infty) = \{-\infty\}\}$.

Definition 2.2.2. [Lur09, Lemma 6.1.3.16]
The *split-simplicial category* $\Delta_{-\infty}$ is defined as follows:

1. Objects: Objects of Δ_+.

2. Morphisms: $\text{Hom}_{\Delta_\infty}([m]_+, [n]_+) = \{\alpha : [m]_+ \to [n]_+ \,,\ \alpha \ \text{non-decreasing} \,;\ \alpha(-\infty) = \infty\}$.

Remark 2.2.3. We list some properties of the categories defined above:

1. The simplicial category Δ (Definition A.1.1) is a full subcategory of Δ_+. For the sake of simplicity, we shall abuse the notation and write $[n]_+$ as $[n]$ by identifying Δ as a full subcategory of Δ_+.

2. The category Δ_+ is formed from Δ by formally adjoining $[-1]$ as the initial object.

3. Let $[m]_+, [n]_+$ be any two objects in $\Delta_{-\infty}$. Then morphisms between $[m]_+ \to [n]_+$ can send elements other than $-\infty$ to $-\infty$. In particular, the unique morphisms $[-1]_+ \to [n]_+$ in Δ_+ admit a section $[n]_+ \to [-1]_+$ in $\Delta_{-\infty}$ given by mapping all elements to $-\infty$.

Notation 2.2.4. The category Δ and Δ_+ has two special collection of maps for every positive integer n.

1. We have $n+1$ maps $d_i^n : [n-1] \to [n]$; $0 \le i \le n$ which are defined as
$$d_i^n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \ge i. \end{cases}$$
These are called *face maps*. The face map $d_i^n : [n-1] \to [n]$ is the unique non-decreasing injective map which does not have i in its image.

2. We have n maps $s_i^{n-1} : [n] \to [n-1]; 0 \le i \le n-1$ which are defined as
$$s_i^{n-1}(j) = \begin{cases} j & \text{if } j \le i \\ j-1 & \text{if } j > i. \end{cases}$$
These are called *degeneracy maps*. The degeneracy map $s_i^n : [n] \to [n-1]$ is the unique non-decreasing surjective map such that the element i has two preimages.

3. The category Δ_+ has the map $d_{-1}^0 : [-1] \to [0]$ which satisfies the relation
$$d_0^1 \circ d_{-1}^0 = d_1^1 \circ d_{-1}^0. \quad \textbf{(D1)}$$

The face and degeneracy maps satisfy the following identites in Δ (also in Δ_+ and $\Delta_{-\infty}$):

1. $d_j^{n+1} \circ d_i^n = d_i^{n+1} \circ d_{j-1}^n \quad 0 \le i < j \le n.$
2. $s_j^n \circ s_i^{n+1} = s_i^n \circ s_{j+1}^{n+1} \quad 0 \le i \le j < n.$
3. $s_j^n \circ d_i^{n+1} = \begin{cases} d_i^n \circ s_{j-1}^{n-1} & 0 \le i < j < n \\ \mathrm{Id}_{[n]} & 0 \le j < n \text{ and } i = j \text{ or } i = j+1 \\ d_{i-1}^n \circ s_j^{n-1} & 0 \le j \text{ and } j+1 < i \le n. \end{cases}$

These relations are called *simplicial identities* and shall be denoted by **S1**.

Proposition 2.2.5. *[Lan71, Page 173] Any morphism* $f : \{0, \cdots, n\} \to \{0, \cdots, n'\}$ *in* Δ *has a unique representation*
$$f = (d_{i_k}^{n'} \circ d_{i_{k-1}}^{n'-1} \circ \cdots \circ d_{j_1}^{n'-h+1}) \circ (s_{j_1}^{n'-h} \circ s_{j_2}^{n'-h-1} \circ \cdots \circ s_{j_h}^{n-1})$$
where the non-negative integers h, k *satisfy* $n + k - h = n'$ *and the subscripts* $i_1, \cdots, i_k$ *and* $j_1, \cdots, j_h$ *satisfy*
$$n' \ge i_k > \cdots i_1 \ge 0 \qquad 0 \le j_1 < \cdots < j_h < n.$$

Corollary 2.2.6. *The relations **S1** provide a presentation of the category* Δ *i.e.* Δ *is the smallest category containing the objects* $[n]$ *and face and degeneracy maps satisfying the relations **S1**.*
*Similarly the relations **S1** and **D1** provide a presentation of the category* Δ_+*.*

Proof. Given any finite composition of face and degeneracy maps, the relation **S1** helps us to rewrite the composition as the one in Proposition 2.2.5. This explains that every morphism in Δ exist in the subcategory generated by face and degeneracy maps subjected to the relation S_1.
The same argument works for the category Δ_+. □

Remark 2.2.7. 1. The skeletal description of Δ looks like:

$$[0] \rightrightarrows [1] \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; [2] \cdots$$

Here the bold arrows are the face maps and dotted arrows are the degeneracy maps.

2. Similarly, the skeletal description of Δ_+ looks like:

$$[-1] \xrightarrow{d^0_{-1}} [0] \rightrightarrows [1] \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; [2] \cdots$$

Notice that the diagram:

$$[-1] \xrightarrow{d^0_{-1}} [0] \underset{d^1_1}{\overset{d^1_0}{\rightrightarrows}} [1]$$

is a fork in the category Δ_+ by the relation **D1**.

In order to get a presentation of the category $\Delta_{-\infty}$, we need to define some additional maps in $\Delta_{-\infty}$.

Notation 2.2.8. The category $\Delta_{-\infty}$ has *splitting maps* for all $i \geq -1$ which are defined as follows:

$$s^i_{-1} : [i+1] \to [i] \ ; \ s^i_{-1}(j) = \begin{cases} -\infty & j = 0, -\infty \\ j-1 & \text{otherwise.} \end{cases}$$

The map s^i_{-1} is the unique non-decreasing surjective map for which $-\infty$ has two preimages.

The maps s^{-1}_i satisfy the following relations:

1. $s^n_{-1} \circ d^{n+1}_0 = \mathrm{id}_{[n]}$ for all $n \geq -1$.
2. $s^n_{-1} \circ d^{n+1}_j = d^n_{j-1} \circ s^{n-1}_{-1} : [n] \to [n]$ where $n \geq 0, 0 < j \leq n+1$.
3. $s^{n-1}_j \circ s^n_{-1} = s^{n-1}_{-1} \circ s^n_{j+1} : [n+1] \to [n-1]$ for all $n > 0, -1 \leq j \leq n-1$.

These relations shall be denoted by **S2**.

Proposition 2.2.9. *Let* $f : [m] \to [n]$ *be a morphism in* $\Delta_{-\infty}$*, then* f *has a unique representation of the form:*

$$f = (d^n_{i_k} \circ d^{n-1}_{i_{k-1}} \circ \cdots \circ d^{n'-h+1}_{j_1}) \circ (s^{n'-h}_{j_1} \circ s^{n'-h-1}_{j_2} \circ \cdots \circ s^{n'-1}_{j_h}) \circ (s^{n'}_{-1} \circ s^{n'-1}_{-1} \cdots \circ s^{m-1}_{-1})$$

where:

1. $n' = m - m'$*. Here* $m' = |\{\ i \in [m] \mid f(i) = -\infty\ ,\ i \neq -\infty\ \}|$.
2. $n' - h + k = n$.

3. *the subscripts* $i_1, \cdots, i_k$ *and* $j_1, \cdots, j_h$ *satisfy*

$$n' \geq i_k > \cdots i_1 \geq 0 \qquad 0 \leq j_1 < \cdots < j_h < n.$$

Proof. Let f be the morphism in $\Delta_{-\infty}$. Notice that $f(j) = -\infty$ for $0 \leq j < m'$. Let $n' = n - m'$. Then f can be factorized as as:

$$[m] \xrightarrow{f'} [n'] \xrightarrow{f''} [n]$$

where:

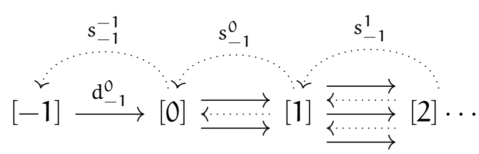
1. $f' = s_{-1}^{n'} \circ s_{-1}^{n'-1} \cdots \circ s_{-1}^{m-1}$.
2. $f'' : [n] \to [n']$ such that $f^{-1}(\{-\infty\}) = \{-\infty\}$.

Now apply the decomposition of f'' considering it as a morphism in Δ_+ by Proposition 2.2.5. This gives us the description of f as required.
As the description of f' is intrinsic to the morphism f, the description of f is unique is equivalent of saying that the description of f'' is unique. This is true because of Proposition 2.2.5. □

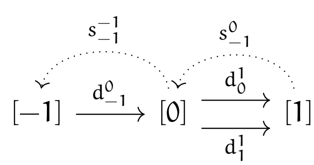
We have the following corollary whose proof is same as the proof of Corollary 2.2.6.

Corollary 2.2.10. *The relations* ***S1***, ***D1*** *and* ***S2*** *provide a presentation of the category* $\Delta_{-\infty}$.

Remark 2.2.11. 1. The skeletal description of $\Delta_{-\infty}$ looks like



2. The diagram



is a split fork. This is because of the relations in **S2** (the first two relations applied to $n = 0$). Hence it is an equalizer in $\Delta_{-\infty}$.

We now recall the definition of simplicial, augmented simplicial and split-simplicial objects.

Definition 2.2.12. [Lur09, Definition 6.1.2.2] Let $\mathcal{C}$ be an ∞-category.

1. A *simplicial object* of $\mathcal{C}$ is a functor $F : N(\Delta)^{op} \to \mathcal{C}$.
2. An *augmented simplicial object* of $\mathcal{C}$ is a functor $F : N(\Delta_+)^{op} \to \mathcal{C}$.
3. A *split-simplicial object* of $\mathcal{C}$ is a functor $F : N(\Delta_{-\infty})^{op} \to \mathcal{C}$.

Remark 2.2.13. 1. Notice that $N(\Delta_+) = N(\Delta)^{\triangleleft}$ (where $(-)^{\triangleleft}$ is the left cone of a simplicial set ([Lur09, Notation 1.2.8.4])). Thus augmented simplicial objects are potential colimit diagrams.

2. Split-simplicial objects are generalized notion of split forks in the simplicial setting (as noted in the previous remark). By the Dold–Kan correspondence, split simplicial objects correspond to augmented exact chain complexes ([Lur17, Section 4.7.2]).

A natural example of a split-simplicial object in explained in the following lemma

Lemma 2.2.14. *Let* C *be an ordinary category admitting products. Let* $f : X \to Y$ *be a morphism in* C *which has a section* $s : Y \to X$ *(i.e.* $f \circ s = \mathrm{id}_Y$*). Then the following diagram*

$$X^+_{\bullet,Y,f,s} := \quad \cdots X \times_Y X \times_Y X \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; X \times_Y X \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; X \longrightarrow Y \tag{2.11}$$

is a split-simplicial object of $N(C)$.

Proof. We need to define a functor $F : N(\Delta_{-\infty})^{op} \to N(C)$. We define it as a functor of the underlying ordinary categories $F : \Delta^{op}_{-\infty} \to C$ as follows:

1. $F([n]) = \begin{cases} Y & \text{if } n = -1 \\ X^n_Y & \text{otherwise} \end{cases}$ where $X^n_Y = \underbrace{X \times X \times_Y X \cdots \times_Y X}_{n+1 \text{ times}}$.

2. Let $p : [m] \to [n]$ be a morphism in $\Delta_{-\infty}$, then

$$F(p) : X^n_Y \to X^m_Y$$

   is defined in the following manner. Then we define the map in the following cases:

   (a) For $n = -1$, $F(p) : Y \to X^m_Y$ given by $y \to (s(y), s(y), \ldots, s(y))$.

   (b) For $m = -1$, $F(p) : X^n_Y \to Y$ given by $(x_0, x_1, ..x_m) \to f(x_0)$. Note that the definition makes sense as $f(x_0) = f(x_i)$ for all i by the property of fiber product.

   (c) For $n, m > -1$, $F(p)(x_0, x_1, ...x_n) = (x'_0, ...x'_m)$ where

$$x'_i = \begin{cases} s \circ f(x_0) & \text{if } p(i) = -\infty \\ x_{p(i)} & \text{otherwise} \end{cases}$$

3. We need to check that the assignment on the level of morphisms is compatible with compositions. This is immediate from the definitions above.

□

Remark 2.2.15. 1. There is another way of proving the lemma above. One can define the morphisms only for splitting, face and degeneracy maps and verify that they satisfy the conditions **S1**,**D1** and **S2**.

2. The above lemma holds true is $N(C)$ is replaced by any ∞-category $\mathcal{C}$ ([Lur17, Proposition 4.7.2.9]).

2.3 Properties of split-simplicial objects.

As mentioned in the beginning of the previous section, split-simplicial objects have the potential to be colimit diagrams. This section quotes the result and ends with a corollary pertaining to the example described in Lemma 2.2.14

Lemma 2.3.1. *[Lur09, Lemma 6.1.3.16] Let $\mathsf{X} : \mathsf{N}(\Delta_{-\infty})^{\mathsf{op}} \to \mathcal{C}$ be a split simplicial object. Then it is a colimit diagram.*

Corollary 2.3.2. *Let $\mathsf{F} : \mathsf{N}(\mathcal{C}) \to \mathcal{D}$ be a functor between ∞-categories. Let $\mathsf{X}^+_{\bullet,\mathsf{Y},\mathsf{f},\mathsf{s}} : \mathsf{N}(\Delta_\infty)^{\mathsf{op}} \to \mathsf{N}(\mathcal{C})$ be the split-simplicial object defined in Lemma 2.2.14.*

Then $\mathsf{F}(\mathsf{Y})$ is the colimit of the diagram

$$\mathsf{F}(\mathsf{X}) \rightleftarrows \mathsf{F}(\mathsf{X} \times_{\mathsf{Y}} \mathsf{X}) \rightrightarrows \cdots$$

Proof. As $\mathsf{X}^+_{\bullet,\mathsf{Y},\mathsf{f},\mathsf{s}}$ is a split-simplicial object in $\mathcal{C}$, then $\mathsf{F}(\mathsf{X}^+_{\bullet,\mathsf{Y},\mathsf{f},\mathsf{s}}) : \mathsf{N}(\Delta_{-\infty})^{\mathsf{op}} \to \mathcal{C} \to \mathcal{D}$ is also a split-simplicial object in $\mathcal{D}$ and thus a colimit diagram by Lemma 2.3.1. □

CHAPTER 3

ENHANCEMENT OF SHEAVES ALONG COVERINGS WITH LOCAL SECTIONS

In this chapter, we prove Theorem 3.4.1 which enables us to extend sheaves from schemes to algebraic stacks. This key point is that for cohomology theories like $\mathcal{SH}$, descent may not be true for general smooth morphisms, which are used as atlases for algebraic stacks. From the previous chapter, we see that descent is automatic for smooth morphisms that admit a section (Lemma 2.2.14). Thus if the atlas admits local sections in a topology that is coarser than the smooth topology, then it is plausible to extend sheaves from schemes to a large class of Artin stacks.

The first section of the chapter introduces the notion of $\mathcal{T}$- local sections associated to a site $(\mathcal{C}, \mathcal{T})$. In the second section, we construct a $(2,1)$-category called "category of stacks admitting $\mathcal{T}$-local sections". In the third section, we introduce the $(2,1)$-category Nis-locSt for which we extend cohomology theories like $\mathcal{SH}$. The abstract formalism of category of stacks admitting $\mathcal{T}$-local sections help us to prove the Theorem 3.4.1 in the fourth section. In the last section, we apply Theorem 3.4.1 to define the stable homotopy theory on Nis-locSt and the derived category of an algebraic stack.

3.1 Morphisms admitting $\mathcal{T}$-local sections.

Let $\mathcal{C}$ be a category admitting products and small coproducts equipped with a Grothendieck topology $\mathcal{T}$. For any $Y \in \mathcal{C}$, let $\mathrm{Cov}(Y)$ be the collection of coverings of Y.

Definition 3.1.1. A morphism $f : X \to Y$ in $\mathcal{C}$ admits *$\mathcal{T}$-local sections* if there exists a family $\{p_i : Y_i \to Y\}_{i \in I} \in \mathrm{Cov}(Y)$ and morphisms $s_i : Y_i \to X$ such that the diagram

$$\begin{array}{ccc} & & X \\ & \overset{\coprod s_i}{\nearrow} & \big\downarrow f \\ \coprod_i Y_i & \xrightarrow{\coprod p_i} & Y \end{array} \tag{3.1}$$

commutes.

Example 3.1.2. In the category of schemes equipped with étale topology, any smooth surjective morphism admits étale local sections. This follows from [Sta21, Tag 055U].

Lemma 3.1.3. *Morphisms admitting $\mathcal{T}$-local sections are stable under pullbacks and compositions.*

Proof. **Stable under pullbacks**: Let $f : X \to Y$ be a morphism admitting $\mathcal{T}$-local section and let $g : Y' \to Y$ be a morphism in $\mathcal{C}$. We denote the pullback of f along g by $f' : X' := X \times_Y Y' \to Y'$. We need to show that f' admits a $\mathcal{T}$-local section.
As f admits $\mathcal{T}$-local sections, there exists a covering $p : \tilde{Y} \to Y$ and a morphism $s : \tilde{Y} \to X$ such that $p = f \circ s$..
Then $p' : \tilde{Y}' := Y' \times_Y \tilde{Y} \to Y'$ is a covering and $s' := \mathrm{id} \times s : \tilde{Y}' \to X'$ satisfies $p' = f' \circ s'$. Thus f' admits $\mathcal{T}$-local sections.

Stable under compositions: Let $f : X \to Y$ and $g : Y \to Z$ be morphisms admitting $\mathcal{T}$-local sections. We need to show that $g \circ f$ admits $\mathcal{T}$-local sections. As g admits $\mathcal{T}$-local sections, we have a covering $q : \tilde{Z} \to Z$ and a section $s : \tilde{Z} \to Y$.
By (1),the morphism $f' := Z' = \tilde{Z} \times_Y X \to \tilde{Z}$ admits $\mathcal{T}$-local sections. So there exists a covering $p : \tilde{Z}'' \to \tilde{Z}$ admitting a section $s'' : \tilde{Z}'' \to \tilde{Z}'$.

These morphisms give rise to a commutative diagram

$$\begin{array}{ccccc} & & & & X \\ & & & \overset{s'}{\nearrow} & \downarrow f \\ & & \tilde{Z}' & & Y \\ & \overset{s''}{\nearrow} & \downarrow f' & \overset{s}{\nearrow} & \downarrow g \\ \tilde{Z}'' & \xrightarrow{p} & \tilde{Z} & \xrightarrow{q} & Z, \end{array} \tag{3.2}$$

i.e.. $s' \circ s''$ is a $\mathcal{T}$-local section of $g \circ f$.

□

Corollary 3.1.4. *The category $\mathcal{C}$ with the set of coverings as*

$$\mathrm{Cov}_{\mathcal{T}\text{-loc}}(X) := \{\{x_i : X_i \to X\} | \coprod_i x_i \textit{ admits } \mathcal{T} - \textit{local sections}\}$$

defines a site.

Proof. By definition, identity morphisms admit $\mathcal{T}$-local sections. As morphisms admitting $\mathcal{T}$-local sections are stable under pullbacks and compositions (Lemma 3.1.3), we see that $\mathcal{C}$ with the coverings $\mathcal{T}$-loc forms a site. □

The following proposition says that sheaves on $\mathcal{C}$ with respect to the topology $\mathcal{T}$ are equivalent as sheaves on $\mathcal{C}$ with respect to the topology $\mathcal{T}$-loc.

Proposition 3.1.5. *Let $F : N(\mathcal{C})^{op} \to \mathcal{D}$ an ∞-sheaf. Then F satisfies descent along morphisms that admit $\mathcal{T}$-local sections.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ which admits $\mathcal{T}$-local sections. Thus there exists a covering $\varphi : \tilde{Y}' \to Y$ and a section $s' : \tilde{Y} \to \tilde{X} := \tilde{Y} \times_Y X$ of $f' : \tilde{X} \to \tilde{Y}$ such that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\varphi'} & X \\ {\scriptstyle s'}\uparrow\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \tilde{Y} & \xrightarrow{\varphi} & Y \end{array}$$

commutes in $\mathcal{C}$.
As f' has a section, f' satisfies F-descent by Corollary 2.3.2.
As F is an ∞-sheaf, the horizontal arrows have F-descent. As f', φ and φ' satisfy F-descent, by Lemma A.16.8 we get that f satisfies F-descent. □

Remark 3.1.6. When $\mathcal{C} = \mathrm{Sch}$ and $\mathcal{T} = \text{ét}$, the above proposition gives us that the category of étale sheaves and category of smooth sheaves are equivalent.

3.2 Categories of stacks admitting $\mathcal{T}$-local sections.

We want to extend sheaves on schemes to the $(2,1)$-category of algebraic stacks. This extension is a two step process. We first extend from schemes to algebraic spaces and then from algebraic spaces to algebraic stacks. In order to formalize the statements in a coherent manner, we define an abstract $(2,1)$-category $\mathrm{St}_{\mathcal{C}}$ which incorporates the properties of algebraic spaces and algebraic stacks.

Definition 3.2.1. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology $\mathcal{T}$. A *category of stacks admitting $\mathcal{T}$-local sections* is a $(2,1)$-category $\mathrm{St}_{\mathcal{C}}$ together with a fully faithful inclusion $i_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathrm{St}_{\mathcal{C}}$ satisfying the following properties:

1. $\mathrm{St}_{\mathcal{C}}$ admits fiber products and small coproducts.

2. Given any object $\mathcal{X} \in \mathrm{St}_{\mathcal{C}}$, there exists a morphism $x : X \to \mathcal{X}$ with $X \in \mathcal{C}$ such that for any morphism $y : X' \to \mathcal{X}$ in $\mathcal{C}$, the fiber product $x' : X' \times_{\mathcal{X}} X \to X'$ admits $\mathcal{T}$-local sections. We say that x as an *atlas admitting $\mathcal{T}$-local sections.*

3. The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable in $\mathcal{C}$.

Remark 3.2.2. Here representablity is understood as for algebraic stacks i.e. a morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathrm{St}_{\mathcal{C}}$ if for all $Y \to \mathcal{Y}$ with $Y \in \mathcal{C}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} Y$ is in $\mathcal{C}$. With this definition, the diagonal map being representable is equivalent of saying that any morphism $x : X \to \mathcal{X}$ where $X \in \mathcal{C}$ is representable.
As in the case of algebraic stacks, representable morphisms are stable under pullbacks.

Definition 3.2.3. Let $\mathrm{St}_{\mathcal{C}}$ be a category of stacks admitting $\mathcal{T}$-local sections.
A morphism $f : \mathcal{X} \to \mathcal{Y}$ is said to admit *$\mathcal{T}$-local sections* if there exists a an atlas $y : Y \to \mathcal{Y}$ and a morphism $s : Y \to \mathcal{X}$ such that $f \circ s = y$.

The following gives a simpler definition for morphisms admitting $\mathcal{T}$-local sections in the setting of representable morphisms.

Lemma 3.2.4. *A representable morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathrm{St}_{\mathcal{C}}$ admits $\mathcal{T}$-local sections iff for any morphism $v : V \to \mathcal{Y}$ with $V \in \mathcal{C}$, the base change morphism $f' : V \times_{\mathcal{Y}} \mathcal{X} \to V$ is a morphism admitting $\mathcal{T}$-local sections in $\mathcal{C}$.*

Proof. Let f be a representable morphism admitting $\mathcal{T}$-local sections. Let $v : V \to \mathcal{Y}$ be a morphism where $V \in \mathcal{C}$. We want to show that the base change morphism $f' : X' := V \times_{\mathcal{Y}} \mathcal{X} \to V$ is a morphism admitting $\mathcal{T}$-local sections. By definition, there exists an atlas admitting $\mathcal{T}$-local sections $v' : V' \to \mathcal{Y}$ and a section $s : V' \to \mathcal{X}$. Then the base change morphism $v'' : V'' := V' \times_{\mathcal{Y}} V \to V$ admits $\mathcal{T}$-local sections.
On the other hand, the section s induces a map $s' : V'' \to X'$. As v'' admits $\mathcal{T}$-local sections, there exists a covering $\tilde{v} : \tilde{V} \to V$ and a section $s'' : \tilde{V} \to V''$.
Thus there exists a covering $\tilde{v}$ and a section $s' \circ s''$ implying that f' admits $\mathcal{T}$-local sections.

For the other direction, let $v : V \to \mathcal{Y}$ be an atlas admitting $\mathcal{T}$-local sections. The assumptions says that the base change morphism $f' : X' := \mathcal{X} \times_{\mathcal{Y}} V$ is a morphism admitting $\mathcal{T}$-local sections. Thus there exists a covering $v' : V' \to V$ and a section $s' : V' \to X'$. Then the compositions $v \circ v'$ and $x' \circ s'$ imply that f admits $\mathcal{T}$-local sections (here x' is the pullback of v along f). ☐

Lemma 3.2.5. *The pullback of an atlas along any morphism in $\mathrm{St}_{\mathcal{C}}$ is a morphism admitting $\mathcal{T}$-local sections.*

Proof. Let $y : Y \to \mathcal{Y}$ be an atlas admitting $\mathcal{T}$-local sections and let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism in $\mathrm{St}_{\mathcal{C}}$. We want to show that $y' : \mathcal{X}' := Y \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ admits $\mathcal{T}$-local sections. As v' is representable, by Lemma 3.2.4 it suffices to show that for any morphism $x : X \to \mathcal{X}$ where $X \in \mathcal{C}$, the fiber product $y'' : X \times_{\mathcal{X}} \mathcal{X}' \to X$ admits $\mathcal{T}$-local sections. This follows from the fact that y'' is pullback of y along $y \circ x$ and y is an atlas. ☐

Lemma 3.2.6. *Morphisms in $\mathrm{St}_{\mathcal{C}}$ admitting $\mathcal{T}$-local sections are stable under pullbacks and compositions.*

Proof. **Stable under pullbacks**: Consider a pullback square

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array} \tag{3.3}$$

where f admits $\mathcal{T}$-local sections. Thus, there exists an atlas $y : Y \to \mathcal{Y}$ and a morphism $g : Y \to \mathcal{X}$ such that $f \circ g = y$. Then the base change morphism $\tilde{y} : \mathcal{Y}'' := Y \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ admits $\mathcal{T}$-local sections. As $\mathcal{Y}'$ admits a morphism to $\mathcal{X}$ (via g), we have a unique morphism $g'' : \mathcal{Y}'' \to \mathcal{X}'$ such that $f' \circ g'' = \tilde{y}$. We denote $y' : Y' \to \mathcal{Y}'' \xrightarrow{\tilde{y}} \mathcal{Y}''$ to be the composition where Y' is an atlas of $\mathcal{Y}''$. Note that y' admits $\mathcal{T}$-local sections. Denote g' to be composition $g' : Y' \to \mathcal{Y}'' \xrightarrow{g''} \mathcal{X}''$. Then we have that $f' \circ g' = y'$ thus proving the fact that f' admits $\mathcal{T}$-local sections.

Stable under compositions: Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms admitting $\mathcal{T}$-local sections. Thus there exists atlases $z : Z \to \mathcal{Z}$, $y : Y \to \mathcal{Y}$ and morphisms $p : Z \to Y$, $q : Y \to \mathcal{X}$ such that $g \circ p = z$ and $f \circ q = y$. Let $p' : Z' := Y \times_{\mathcal{Y}} Z \to Y$ and $y' : Z' \to Z$ be the base change of p and y respectively. Thus y' admits $\mathcal{T}$-local sections. We denote the compositions by $z' := z \circ y' : Z' \to \mathcal{Z}$ and $q' := q \circ p' : Z' \to \mathcal{X}$. Thus we get that $q' \circ (g \circ f) = z'$ implying that $g \circ f$ admits $\mathcal{T}$-local sections (as z' is an atlas of $\mathcal{Z}$).

□

The following lemma gives us another definition of morphisms admitting $\mathcal{T}$-local sections which shall help us to prove the sheaf condition in Theorem 3.4.1.

Lemma 3.2.7. *A morphism $f : \mathcal{X} \to \mathcal{Y}$ admits $\mathcal{T}$-local sections iff there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{x} & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{y} & \mathcal{Y} \end{array} \tag{3.4}$$

where f' admits $\mathcal{T}$-local sections in $\mathcal{C}$ and x, y are atlases admitting $\mathcal{T}$-local sections.

Proof. Let f be a morphism admitting $\mathcal{T}$-local sections. Let $y : Y \to \mathcal{Y}$ be an atlas admitting $\mathcal{T}$-local sections. Then the base change morphisms $f'' : \mathcal{X}' := \mathcal{X} \times_{\mathcal{Y}} Y \to Y$ and $x' : \mathcal{X}' \to \mathcal{X}$ admit $\mathcal{T}$-local sections. Let $x'' : X \to \mathcal{X}'$ be an atlas of $\mathcal{X}'$. Then the compositions $f' := f'' \circ x''$ and $x := x' \circ x''$ admit $\mathcal{T}$-local sections giving us the commutative diagram that was needed. For the other direction, consider a commutative square

$$\begin{array}{ccc} X & \xrightarrow{x} & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{y} & \mathcal{Y} \end{array} \tag{3.5}$$

where f' admits $\mathcal{T}$-local sections and x, y are atlases. By definition there exists a covering $y' : Y' \to Y$ in the topology $\mathcal{T}$ and a morphism $s' : Y' \to X$ such that $f \circ s = y'$. Defining $s = x \circ s'$ and $y'' : Y' \to Y \to \mathcal{Y}$ we get that $f \circ s = y''$ where y'' is an atlas of $\mathcal{Y}$. Hence f admits $\mathcal{T}$-local sections.

□

For any object $\mathcal{X} \in \mathrm{St}_{\mathcal{C}}$, define $\mathrm{Cov}(\mathcal{X})$ as the set of families of the form $\{x_i : \mathcal{X}_i \to \mathcal{X}\}_{i \in I}$ such that $x := \coprod_i x_i : \coprod_i \mathcal{X}_i \to \mathcal{X}$ admits $\mathcal{T}$-local sections.

Lemma 3.2.8. *The family of coverings admitting $\mathcal{T}$-local sections* $\mathrm{Cov}(\mathcal{X})$ *for every object $\mathcal{X} \in \mathrm{St}_{\mathcal{C}}$ defines a Grothendieck topology on* $\mathrm{St}_{\mathcal{C}}$*. We will write* $(\mathrm{St}_{\mathcal{C}}, \mathcal{T}\text{-}\mathrm{loc})$ *for the corresponding site.*

Proof. The identity morphisms in $\mathrm{St}_{\mathcal{C}}$ admit $\mathcal{T}$-local sections. By Lemma 3.2.6, $\mathcal{T}$-local sections are stable under pullbacks and compositions. Thus the $(2,1)$-category $\mathrm{St}_{\mathcal{C}}$ defines a site.

□

3.3 The $(2,1)$-category Nis-locSt.

In this section, we introduce the class of stacks for which we can extend cohomology theories that satisfy descent with respect to the Nisnevich topology. This class of stacks will be called Nis-locSt and we will explain that many interesting Artin stacks are contained in this class. In particular, we show that this contains all local quotient stacks and quasi-separated algebraic spaces.

At first, we consider $\mathcal{C} = \mathrm{Sch}$, the category of schemes equipped with the Nisnevich topology. Then any quasi-separated algebraic space has an atlas admitting Nisnevich-local sections ([Knu71, Chapter 2, Theorem 6.3]). So we can consider $\mathrm{St}_{\mathcal{C}} = \mathrm{N}(\mathrm{Algsp})$ to be the category of quasi-separated algebraic spaces.

Remark 3.3.1. By the above discussion, given any quasi-separated algebraic space $\mathcal{X}$, there exists a Nisnevich covering $x : X \to \mathcal{X}$ where X is a scheme.

Notation 3.3.2. The category Nis-locSt is the category of algebraic stacks for which there exists a smooth atlas admitting Nisnevich-local sections. In the terminology introduced in Definition 3.2.1, this is the category of stacks admitting Nisnevich-local sections for the category $\mathrm{N}(\mathrm{Algsp})$ of quasiseparated algebraic spaces.

Before listing some examples of algebraic stacks in Nis-locSt, let us verify some properties of the category Nis-locSt.

Lemma 3.3.3. *The* $(2,1)$*-category* Nis-locSt *admits fiber products.*

Proof. Let

$$\begin{array}{ccc} & & \mathcal{X}_2 \\ & & \downarrow \\ \mathcal{X}_1 & \longrightarrow & \mathcal{X}_0 \end{array} \tag{3.6}$$

be a diagram where $\mathcal{X}_0, \mathcal{X}_1$ and $\mathcal{X}_2$ are objects in Nis-locSt. We want to show that $\mathcal{X}_3 := \mathcal{X}_1 \times_{\mathcal{X}_0} \mathcal{X}_2$ is algebraic stack which has an atlas admitting Nisnevich-local sections.
Let $x_0 : X_0 \to \mathcal{X}_0$ be an atlas admitting Nisnevich-local sections. At first, we prove the following claim

Claim 3.3.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism in Nis-locSt, then for every atlas $y : Y \to Y$, there exists of a 2-commutative square

$$\begin{array}{ccc} X & \xrightarrow{x} & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{y} & \mathcal{Y} \end{array} \tag{3.7}$$

where f' is a morphism in $\mathcal{C}$ and x is an atlas admitting Nisnevich-local sections.

Proof of claim. Let $x : X_0 \to \mathcal{X}$ be an atlas of $\mathcal{X}$. Then the base change $x' : X' := \mathcal{X} \times_{\mathcal{Y}} X \to \mathcal{X}$ is a morphism admitting $\mathcal{T}$-local sections. Therefore $x : X := X_0 \times_{\mathcal{X}} X' \to X' \xrightarrow{x'} \mathcal{X}$ is an atlas of $\mathcal{X}$ fitting in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\tilde{y}} & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{y} & \mathcal{Y} \end{array} \tag{3.8}$$

where f lies in $\mathcal{C}$ and y, x are morphisms admitting $\mathcal{T}$-local sections.

□

Applying the claim, we get that there exist atlases $x_1 : X_1 \to \mathcal{X}_1$ and $x_2 : X_2 \to \mathcal{X}_2$ such that the following diagrams

$$\begin{array}{ccc} X_1 & \longrightarrow & X_0 \\ \downarrow{\scriptstyle x_1} & & \downarrow{\scriptstyle x_0} \\ \mathcal{X}_1 & \longrightarrow & \mathcal{X}_0, \end{array} \quad \begin{array}{ccc} X_2 & \longrightarrow & X_0 \\ \downarrow{\scriptstyle x_2} & & \downarrow{\scriptstyle x_0} \\ \mathcal{X}_2 & \longrightarrow & \mathcal{X}_0, \end{array} \tag{3.9}$$

commute. This induces a natural map $x_3 : X_3 := X_1 \times_{X_0} X_2 \to \mathcal{X}_3$. We claim that x_3 is an atlas admitting Nisnevich-local sections. Let $v_3 : V \to \mathcal{X}_3$ be a morphism where V is a scheme. Then it induces maps $v_1 : V \to \mathcal{X}_1$, $v_0 : V \to \mathcal{X}_0$ and $v_2 : V \to \mathcal{X}_2$. Thus the base change morphisms $x_1' : X_1' := V \times_{\mathcal{X}_1} X_1 \to V$, $x_2' : X_2' := V \times_{\mathcal{X}_2} X_2 \to V$ and $x_0' : X_0' := V \times_{\mathcal{X}_0} X_0 \to V$ admit Nisnevich-local sections. As the fiber product $V \times_{\mathcal{X}_3} X_3 \cong X_1' \times_{X_0'} X_2'$, the morphism $x_3' : V \times_{\mathcal{X}_3} X_3 \to V$ admits Nisnevich-local sections. Thus x_3 is an atlas admitting Nisnevich-local sections. □

Lemma 3.3.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces such that $\mathcal{Y} \in$ Nis-locSt, then $\mathcal{X} \in$ Nis-locSt.*

Proof. Let $y : Y \to \mathcal{Y}$ be an atlas admitting Nisnevich local sections. By Remark 3.3.1, it suffices to show that the base change morphism $x : X := Y \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ admits Nisnevich-local sections where X is an algebraic space. Let $x' : X' \to \mathcal{X}$ be a morphism where X is a scheme. As $X'' := X' \times_{\mathcal{X}} X \cong X' \times_{\mathcal{Y}} Y$, the base change morphism $y' : X'' \to Y$ is a morphism of scheme admitting Nisnevich-local sections. □

Before proving the next corollary, let us recall that an algebraic stack $\mathcal{X}$ is a local quotient stack if it admits a open covering by quotient stacks of the form $[X/G]$ where G is an affine algebraic group ([FHT11, A.2.2]).

Corollary 3.3.6. *All local quotient stacks are contained in* Nis-locSt.

Proof. As Zariski open coverings admit Zariski-local sections, it suffices to prove that quotient stacks lie in Nis-locSt. At first, we see that BGL_n lies in Nis-locSt. This is because the atlas $pt \to BGL_n$ is a GL_n-torsor. As GL_n is special, GL_n-torsors are Zariski-locally trivial.
If G is an affine algebraic group, then the inclusion $i : G \hookrightarrow GL_n$ induces a representable morphism $i : BG \to BGL_n$. As $BGL_n \in$ Nis-locSt, by Lemma 3.3.5 we get that $BG \in$ Nis-locSt. The map $[X/G] \to BG$ is representable. Applying Lemma 3.3.5, we get that $[X/G] \in$ Nis-locSt.

□

Remark 3.3.7. Note that unless G is a special group, the standard atlas $X \to [X/G]$ of a quotient stack may not admit Nisnevich-local sections. In the proof above, this atlas is replaced by a scheme X'' which is a Nisnevich cover of the algebraic space $X \times^G \mathrm{GL}_n$. Lets us explain this in detail.
We can write $[X/G]$ as $[X \times^G \mathrm{GL}_n/\mathrm{GL}_n]$. The object $X \times^G \mathrm{GL}_n$ exists as an algebraic space. Thus the morphism $x' : X \times^G \mathrm{GL}_n \to [X/G]$ is a GL_n-torsor and hence admits Nisnevich-local sections. By Remark 3.3.1, we get that there exists a Nisnevich cover $x'' : X'' \to X \times^G \mathrm{GL}_n$ where X'' is a scheme. Hence, we get that the morphism $x' \circ x'' : X'' \to [X/G]$ is an atlas admitting Nisnevich-local sections.

Recall that Totaro and Gross explained that the property of being a quotient stack is closely related to the resolution property ([Tot04] and[Gro17]).

Corollary 3.3.8. *Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack which has affine stabilizers at closed points and satisfies the resolution property. Then $\mathcal{X} \in$ Nis-locSt.*

Proof. Under the assumptions, we get that $\mathcal{X} \cong [U/\mathrm{GL}_n]$ where U is a quasi-affine scheme ([Tot04, Theorem 1.1] and [Gro17, Theorem A]). Thus Corollary 3.3.6 implies that $\mathcal{X} \in$ Nis-locSt. □

We now explain how local constructions like blow ups and deformation to normal cone ([LMB00, Chapter 14]) also lie in Nis-locSt. Before stating the corollary, let us briefly recall the notions. Let $\mathcal{X}$ be an algebraic stacks and $z : \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack and $\mathscr{I} \subset \mathcal{O}_{\mathcal{X}}$ be the ideal sheaf of $\mathcal{Z}$. For any scheme T, we shall denote T' to be the fiber product of T along z.

Notation 3.3.9.

1. The blowup of $\mathcal{X}$ along $\mathcal{Z}$ is the algebraic stack $\mathrm{Bl}_{\mathcal{Z}}(\mathcal{X}) := \mathrm{Proj}(\oplus_{n\geq 0}\mathscr{I}^n_{\mathcal{Z}})$ which admits a morphism representable by schemes $\mathrm{pr}_{\mathrm{bl}} : \mathrm{Bl}_{\mathcal{Z}}(\mathcal{X}) \to \mathcal{X}$ such that for any morphism $T \to \mathcal{X}$, the fiber product $\mathrm{Bl}_{\mathcal{Z}}(\mathcal{X}) \times_{\mathcal{X}} T$ is isomorphic to $\mathrm{Bl}_{T'} T$.

2. The normal cone $N_{\mathcal{Z}}(\mathcal{X}) := \mathrm{Spec}(\oplus_{n\geq 0}\mathscr{I}^n/\mathscr{I}^{n+1})$ is the algebraic stack which admits a morphism $\mathrm{pr}_n : N_{\mathcal{Z}}(\mathcal{X}) \to \mathcal{Z}$ representable by schemes.

3. The deformation to the normal cone $D_{\mathcal{Z}}(\mathcal{X})$ is the analog of deformation space in the setting of schemes. Let us recall the definition in the setting of schemes ([Ful84, Chapter 6]). Let X be a scheme and Z be a closed subscheme of X. Then the deformation space is defined as
$$D_Z X := \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbf{A}^1)/\,\mathrm{Bl}_{Z\times\{0\}}(X \times \{0\}).$$
The algebraic stack $D_{\mathcal{Z}}(\mathcal{X})$ admits a schematic representable morphism $\mathrm{pr}_d : D_{\mathcal{Z}}(\mathcal{X}) \to \mathcal{X} \times \mathbf{A}^1$ such that the fibers of pr_d over $\mathcal{X} \times \{0\}$ and $\mathcal{X} \times \{1\}$ are the normal cone $N_{\mathcal{Z}}(\mathcal{X})$ and the stack $\mathcal{X}$ respectively.

Corollary 3.3.10. *Let $\mathcal{X} \in$ Nis-locSt and let $\mathcal{Z}$ be a closed substack of $\mathcal{X}$. Then $\mathrm{Bl}_{\mathcal{Z}}(\mathcal{X}), N_{\mathcal{Z}}(\mathcal{X})$ and $D_{\mathcal{Z}}(\mathcal{X})$ belong to Nis-locSt.*

Proof. As the morphisms $\mathrm{pr}_{\mathrm{bl}} : \mathrm{Bl}_{\mathcal{Z}}(\mathcal{X}) \to \mathcal{X}, \mathrm{pr}_n : N_{\mathcal{Z}}(\mathcal{X}) \to \mathcal{Z}, \mathrm{pr}_d : D_{\mathcal{Z}}(\mathcal{X}) \to (\mathcal{X} \times \mathbf{A}^1)$ are representable, Lemma 3.3.5 gives us that these algebraic stacks also lie in Nis-locSt. □

Corollary 3.3.11. *1. For any projective variety* X*, the stack of vector bundles* Bun_n *and the stack of* G*-bundles* Bun_G *for an affine algebraic group* G *are in* Nis-locSt*. The same result holds for stacks of Higgs bundles* Higgs_G*.*

2. The moduli spaces of stable maps are in Nis-locSt*.*

Proof. 1. The stack of vector bundles Bun_n can be written as a union of $\mathrm{Bun}_n^{\leq m}$ where Bun_n^m is the open substack of vector bundles of bounded maximal slope m. The stack Bun_n^m is a locally closed substack of a quotient stack by Quot scheme construction ([HL10, Theorem 3.3.7 and Section 4.3]). Thus Bun_n is a local quotient stack and hence by Corollary 3.3.6 lies in Nis-locSt.

As the morphism $\mathrm{Bun}_G \to \mathrm{Bun}_n$ is representable, we get that Bun_G is in Nis-locSt (Lemma 3.3.5). The same argument holds for Higgs bundles as $\mathrm{Higgs}_G \to \mathrm{Bun}_G$ is representable.

2. The moduli space of stable maps is isomorphic to a quotient stack of the form $[J/\,\mathrm{PGL}_n]$ where J is a quasi-projective variety([FP97, Section 2.4]). Hence by Corollary 3.3.6, the moduli space of stable maps lies in Nis-locSt.

□

Remark 3.3.12. In some applications, it is useful to use other topologies other than the Nisnevich topology, for example

1. Consider the category of schemes Sch with the étale topology. Then by similar reasoning in the example before, one can consider $\mathrm{St}_{\mathcal{C}} = N(\mathrm{Algsp})$. Now considering $\mathcal{C} = N(\mathrm{Algsp})$ with the étale topology, one can consider $\mathrm{St}_{\mathcal{C}}$ to be the $(2,1)$-category of all algebraic stacks which we denote by AlgSt. This follows from Example 3.1.2.

2. We could also use the Zariski topology. This class no longer includes algebraic spaces, but still contains many interesting stacks:

   (a) Quotient stacks $[X/G]$ where G is special.

   (b) Quotient stacks $[X/G]$ where X is quasi-projective with a G-linearized action.

   (c) Quotient stacks $[X/G]$ where G is connected and X is equivariantly embedded as a closed subscheme of a normal variety.

   When G is special, the atlas $X \to [X/G]$ is a G-torsor. As G is special, a G-torsor is Zariski-locally trivial. Hence, it is a Zariski-local section.

   In the other cases, by [EG98, Proposition 23] to $U = \mathrm{GL}_n$, the quotient $X \times^G \mathrm{GL}_n$ exists as a scheme. Thus, we get $[X/G] = [X \times^G \mathrm{GL}_n \,/\, \mathrm{GL}_n]$.

3.4 Extension of sheaves from schemes to algebraic stacks.

In this section, we state and prove the theorem which helps us to extend ∞-sheaves from schemes to algebraic stacks. As we will use Čech nerves to verify the sheaf condition, we will from now on assume that the categories $\mathcal{C}$ and $\mathrm{St}_{\mathcal{C}}$ satisfies the conditions in Proposition A.17.7 i.e. coproducts are disjoint and finite coproducts are universal. These conditions are satisfied in all of the examples in the previous section. Theorem 3.4.1 is a special case of [LZ17, Proposition 4.1.1]. As this result is crucial for our construction of $\mathcal{SH}^{\otimes}_{\text{ext}}(-)$ and the special case allows for a shorter proof, we give a self-contained proof of the theorem. Before formulating the result, let us recall that an ∞-category $\mathcal{D}$ admits geometric realizations if any simplicial object of $\mathcal{D}$ admits a colimit in $\mathcal{D}$.

Theorem 3.4.1. *Let $(\mathsf{C}, \mathcal{T})$ be a site and $\mathrm{St}_{\mathcal{C}}$ a category of stacks admitting $\mathcal{T}$-local sections. Let $\mathsf{F} : \mathsf{N}(\mathcal{C}^{\mathsf{op}}) \to \mathcal{D}$ be an ∞-sheaf where $\mathcal{D}^{\mathsf{op}}$ is an ∞-category admitting geometric realizations. Then F can be extended to an ∞-sheaf $\mathsf{F}_{\mathsf{ext}}$ on $(\mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}}), \mathcal{T}\text{-loc})$.*

In particular given any object $\mathcal{X} \in \mathrm{St}_{\mathcal{C}}$ and an atlas $x : X \to \mathcal{X}$ admitting $\mathcal{T}$-local sections, $\mathsf{F}_{\mathsf{ext}}(\mathcal{X})$ can be computed as a limit over the Čech nerve $X^{+}_{\bullet,x}$ over x. In other words,

$$\mathsf{F}_{\mathsf{ext}}(\mathcal{X}) \cong \lim(\ \mathsf{F}(X) \rightrightarrows \mathsf{F}(X \times_{\mathcal{X}} X) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots \ ). \tag{3.10}$$

Idea of constructing the functor $\mathsf{F}_{\mathsf{ext}}$: Given any zero simplex σ_0 i.e. an object $\mathcal{X} \in \mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}})$, we would like to define $\mathsf{F}_{\mathsf{ext}}(\mathcal{X})$ by Eq. (3.10). As this definition depends on the atlas x, we start with an intrinsic description considering all Čech nerves of atlases of objects of $\mathrm{St}_{\mathcal{C}}$.
Define $\mathrm{Cov}(\mathrm{St}_{\mathcal{C}})$ to be the Duskin nerve of the subcategory of the $(2,1)$-category $\mathrm{Fun}((\Delta_{+})^{\mathsf{op}}, \mathrm{St}_{\mathcal{C}})$ whose objects are Čech nerves $\sigma : \mathsf{N}(\Delta_{+})^{\mathsf{op}} \to \mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}})$ of atlases admitting $\mathcal{T}$-local sections of objects of $\mathrm{St}_{\mathcal{C}}$. We shall denote the objects of $\mathrm{Cov}(\mathrm{St}_{\mathcal{C}})$ by pairs $(\mathcal{X}, x : X \to \mathcal{X})$ where $\mathcal{X} \in \mathrm{St}_{\mathcal{C}}$ and $x : X \to \mathcal{X}$ is an atlas admitting $\mathcal{T}$-local sections.

The inclusion $[-1] \hookrightarrow \mathsf{N}(\Delta_{+})$ induces the morphism

$$p : \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})^{\mathsf{op}} \to \mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}})^{\mathsf{op}}.$$

As every object in $\mathrm{St}_{\mathcal{C}}$ admits a cover, the morphism p is surjective on the level of objects.

Claim 3.4.2. The morphism $p : \mathrm{Cov}(\mathrm{St}_{\mathcal{C}}) \to \mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}})$ is surjective on n-simplicies. More precisely, let σ_n be an n-simplex of $\mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}})$ where $n \geq 1$. Then there exists a map

$$\sigma^1_n : \Delta^1 \times \Delta^n \to \mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}})$$

such that

1. $\sigma^1_n|_{[0]\times\Delta^n}$ factors through $\mathsf{N}(\mathcal{C}) \subset \mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}})$,
2. $\sigma^1_n([1] \times \Delta^n) = \sigma_n$ and
3. $\sigma^1_n(\Delta^1 \times [j])$ is a morphism admitting $\mathcal{T}$-local sections for all $0 \leq j \leq n$.

Proof of the claim. The case $n = 1$ follows from Claim 3.3.4. The general case follows by induction as for any n-simplex $(\mathcal{X}_0, \dots, \mathcal{X}_n)$, there exists compatible choice of atlas $(X_0, \dots, X_{n-1})$ by induction. Such a family can be extended to a compatible family $(X_0, X_1, \dots, X_n)$.

This proves the surjectivity of p on the level of simplices because the Čech nerve of σ^1_n (considered as an edge $\Delta^1 \to \mathrm{Fun}(\Delta^n, \mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}}))$) produces an element in n-simplex of $\mathrm{Cov}(\mathrm{St}_{\mathcal{C}})$. □

Morphisms of coverings induce morphisms of Čech nerves that are mapped to the identity via p. We shall denote the collection of all these morphisms in $\mathrm{Cov}(\mathrm{St}_{\mathcal{C}})$ by R. These are called *refinements of coverings*.

Proposition 3.4.3. *The morphism* $p : \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})^{op} \to N^D_\bullet(\mathrm{St}_{\mathcal{C}})^{op}$ *is a localization of* $\mathrm{Cov}(\mathrm{St}_{\mathcal{C}})$ *along* R.

Proof. As p sends R to equivalences, the morphism p induces a morphism

$$p' : \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})[R^{-1}]^{op} \to N^D_\bullet(\mathrm{St}_{\mathcal{C}})^{op}$$

where $i : \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})^{op} \to \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})^{op}[R^{-1}]$ is the anodyne map constructed in existence of localization (Remark A.14.4). By Proposition A.14.3, we want to show that p' is a categorical equivalence. In particular we show that p' is a trivial fibration of simplicial sets (which is a categorical equivalence by Lemma A.8.2).
Thus given any commutative diagram of simplicial sets

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\tau_n} & \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})[R^{-1}]^{op} \\ \big\downarrow & \overset{\tau'_n}{\nearrow} & \big\downarrow p' \\ \Delta^n & \xrightarrow{\sigma_n} & N^D_\bullet(\mathrm{St}_{\mathcal{C}})^{op}, \end{array} \tag{3.11}$$

we need to show the existence of a dotted arrow such that the diagram commutes.
As the objects of $\mathrm{Cov}(\mathrm{St}_{\mathcal{C}})[R^{-1}]^{op}$ and $\mathrm{Cov}(\mathrm{St}_{\mathcal{C}})^{op}$ coincide and p is surjective on 0-simplices, this implies that p' is surjective on 0-simplices. This shows the claim for $n = 0$.

Let $n \geq 1$. We shall denote the vertices of σ_n and τ_n by $\mathcal{X}_0, \mathcal{X}_1, \cdots \mathcal{X}_n$ and $(\mathcal{X}_0, x_0 : X_0 \to \mathcal{X}), (\mathcal{X}_1, x_1 : X_1 \to \mathcal{X}_1), \cdots (\mathcal{X}_n, x_n : X_n \to \mathcal{X}_n)$. As p is surjective on n-simplices, there exists a morphism $\sigma'_n : \Delta^n \to \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})^{op}$ which lifts σ_n. Let us denote the vertices of σ'_n by $(\mathcal{X}_0, x'_0 : X'_0 \to \mathcal{X}_0), (\mathcal{X}_1, x'_1 : \mathcal{X}'_1 \to \mathcal{X}_1), \ldots, (\mathcal{X}_n, x'_n : X'_n \to \mathcal{X}_n)$.
For each $0 \leq i \leq n$, the morphism $x''_i : X''_i := X_i \times_{\mathcal{X}_i} X'_i \to \mathcal{X}_i$ is an atlas admitting $\mathcal{T}$-local sections. The morphisms σ'_n and τ_n induces a morphism

$$\sigma''_n : \partial\Delta^n \to \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})[R^{-1}]^{op}$$

whose vertices are given by $(\mathcal{X}_0, x''_0), (\mathcal{X}_1, x''_i), \ldots, (\mathcal{X}_n, x''_n)$. Note that the projection maps $\mathrm{pr}_i : X''_i \to X_i$ and $\mathrm{pr}'_i : X''_i \to \mathcal{X}'_i$ are elements of R and therefore become equivalences in the localization. This induces a map

$$f_n : \partial\Delta^n \times \Delta^1 \coprod_{\{0\}\times\partial\Delta^n} \{0\} \times \Delta^n \to \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})[R^{-1}]^{op}$$

where $f_n|_{\{0\}\times\Delta^n} = \sigma'_n$, $f_n|_{\{1\}\times\partial\Delta^n} = \sigma''_n$ and $f_n|_{[k]\times\Delta^1} = \mathrm{pr}'_k$ for all $0 \leq k \leq n$. Applying Lemma A.8.14 to the morphism f_n induces a morphism $f'_n : \Delta^n \times \Delta^1 \to \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})[R^{-1}]^{op}$. In particular the morphism σ''_n extends to a morphism $\tau''_n : \Delta^n \to \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})[R^{-1}]^{op}$. The morphisms τ''_n and τ_n produces a map

$$g_n : \partial\Delta^n \times \Delta^1 \coprod_{\partial\Delta^n\times\{1\}} \Delta^n \times \{1\} \to \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})[R^{-1}]^{op}$$

where $g_n|_{\Delta^n \times \{1\}} = \tau''_n$, $g_n|_{\partial\Delta^n \times \{0\}} = \tau_n$ and $g_n|_{[k]\times\Delta^1} = \mathrm{pr}_k$ for $0 \leq k \leq n$. As the morphism $\mathrm{Fun}(\Delta^n, \mathcal{D}) \to \mathrm{Fun}(\partial\Delta^n, \mathcal{D})$ is an isofibration for $n \geq 1$,([Lan21, Proposition 2.2.5], see Lemma A.8.14), g_n extends to a morphism $g'_n : \Delta^n \times \Delta^1 \to \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})[R^{-1}]^{\mathrm{op}}$. In particular, we have extended τ_n to a morphism $\tau'_n : \Delta^n \to \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})[R^{-1}]^{\mathrm{op}}$. Thus there exists a solution to the lifting problem. This shows that p' is a trivial fibration. □

The fact that the morphism p is a localization makes it easy to construct the extension F_{ext} in Theorem 3.4.1 by first extending F to the Čech nerves of coverings and notion that the sheaf consition implies that this induces a functor on the localization $\mathrm{Cov}[R^{-1}]$. Let us explain this in detail.

Proof of Theorem 3.4.1. 1. (**Constructing the functor** F_{ext})

We define a morphism

$$\phi : \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})^{\mathrm{op}} \xrightarrow{F} \mathrm{Fun}(N(\Delta), \mathcal{D}) \xrightarrow{i} \mathrm{Fun}(N(\Delta_+), \mathcal{D}) \xrightarrow{\mathrm{res}|_{[-1]}} \mathcal{D}$$

as follows:

(a) The map
$$F : \mathrm{Cov}(\mathrm{St}_{\mathcal{C}})^{\mathrm{op}} \to \mathrm{Fun}(N(\Delta), \mathcal{D})$$
is the functor F applied to the restricted simplicial object $X_{\bullet,x}$ of an object $X^+_{\bullet,x}$ of $\mathrm{Cov}(\sigma_0)$.

(b) To associate limit diagrams to cosimplicial objects, we apply [Lur09, Corollary 4.3.2.16] which we recall in Proposition A.11.6. Let $\mathcal{C}^{(0)} = N(\Delta)$ and $\mathcal{C} = N(\Delta_+)$. As $\mathcal{D}$ admits geometric realizations, we can apply Proposition A.11.6 to get a morphism
$$i : \mathrm{Fun}(N(\Delta), \mathcal{D}) \to \mathrm{Fun}(N(\Delta_+), \mathcal{D}).$$
On the level of objects, the morphism i sends a cosimplicial object to an augmented cosimplicial object given by its limit diagram (Remark A.11.7).

(c) The map
$$\mathrm{res}|_{[-1]} : \mathrm{Fun}(N(\Delta_+), \mathcal{D}) \to \mathcal{D}$$
is induced by the inclusion map $[-1] \to \Delta_+$. On the level of objects, it sends an augmented cosimplicial object $Y^+_\bullet$ to Y_{-1}.

For any object $X^+_{\bullet,x}$ in $\mathrm{Cov}(\mathrm{St}_{\mathcal{C}})$, we claim that $\phi(X^+_{\bullet,x}) \simeq \lim_{\bullet\in\Delta} F(X_{\bullet,x})$. This follows from the sheaf condition, namely let $X^+_{\bullet,x}$ and $X'^+_{\bullet,x}$ be two Čech nerves of two atlases $x : X \to \mathcal{X}$ and $x' : X \to \mathcal{X}$. Applying Lemma A.16.8 to the pullback square

$$\begin{array}{ccc} X \times_{\mathcal{X}} X' & \longrightarrow & X \\ \downarrow & & \downarrow x \\ X' & \xrightarrow{x'} & \mathcal{X}, \end{array} \tag{3.12}$$

we get that the morphisms $\phi(X^+_{\bullet,x}) \to \phi(X^+_{\bullet,x} \times X'^+_{\bullet,x})$ and $\phi(X^+_{\bullet,x}) \to \phi(X^+_{\bullet,x} \times X'^+_{\bullet,x})$ are equivalences because F satisfies descent along morphisms admitting $\mathcal{T}$-local sections

(Proposition 3.1.5). Morever if $f : (\mathcal{X}, x : X \to \mathcal{X}) \to (\mathcal{X}, x' : X' \to \mathcal{X})$ is a morphism in $\mathrm{Cov}(\mathrm{St}_{\mathcal{C}})$, then the above argument shows us that ϕ sends f to equivalences. Thus ϕ maps every element of R to equivalences. As the morphism p is a localization (Proposition 3.4.3), there exists a functor

$$F_{ext} : N^{D}_{\bullet}(\mathrm{St}_{\mathcal{C}})^{op} \to \mathcal{D}$$

such that $\phi \cong F_{ext} \circ p$ in $\mathrm{Fun}(\mathrm{Cov}(\mathrm{St}_{\mathcal{C}})^{op}, \mathcal{D})$. The equivalence $\phi \circ F_{ext} \circ p$ gives us that $F_{ext}(\mathcal{X})$ can be computed as a simplicial limit over any Čech cover of an atlas admitting $\mathcal{T}$-local sections.

2. (**F_{ext} is an ∞-sheaf**) For showing that F_{ext} is an ∞-sheaf, we need to check that for any morphism $p : \mathcal{Y} \to \mathcal{X}$ admitting $\mathcal{T}$-local sections, p satisfies F_{ext}-descent. Thus we want to show that
$$F_{ext}(\mathcal{X}) \cong \lim_{n \in \Delta} \mathcal{F}_{ext}(X^{+}_{\bullet,p}).$$
As p admits $\mathcal{T}$-local sections, there exists a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{q'} & \mathcal{Y} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{q} & \mathcal{X} \end{array}$$

where p' admits $\mathcal{T}$-local sections and q', q' are atlases admitting $\mathcal{T}$-local sections (Lemma 3.2.7). By assumption, p' satisfies F_{ext}-descent. Also q and q' satisfy F_{ext}-descent by definition of the functor F_{ext}. Then applying Lemma A.16.8, we get that p satisfies F_{ext}-descent. This completes the proof.

□

3.5 Defining SH and derived categories of ℓ-adic sheaves for algebraic stacks.

In this section, we apply Theorem 3.4.1 to extend the definition of $\mathcal{SH}(-)$ (Definition C.3.1) from the level of schemes to algebraic stacks. To illustrate the idea, we also explain how the same theorem allows to extend the definition of derived ∞-category of ℓ-adic sheaves from schemes to stacks as explained in [LZ17].

3.5.1 The Derived ∞-category of ℓ-adic sheaves of an algebraic stack.

Let X be a scheme. There is a good notion of a derived category of ℓ-adic sheaves $D_{et}(X, \mathbf{Q}_l)$ on a scheme X ([BS15]). Enhancing the notion of derived categories to the level of ∞-categories ([Lur17, Section 1.3]), one gets an étale sheaf

$$\mathcal{D}_{et}(-, \mathbf{Q}_l) : N(\mathrm{Sch})^{op} \to \mathrm{Pr}^{L}_{stb}$$

where the functor takes value in the ∞-category of presentable and stable ∞-categories. Let us recall briefly the additional structure encoded in the statement that the functor takes

values in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}}$. Stable ∞-categories take the role triangulated categories, presentability is a finiteness condition that is needed to apply the adjoint functor theorem and the superscript L encodes the property that the pullback functors f^* is colimit preserving.
We get the following corollary.

Corollary 3.5.1. *The functor $\mathcal{D}_{et}(-, \mathbf{Q}_l)$ extends to an ∞-sheaf*

$$\mathcal{D}_{et}(-, \mathbf{Q}_l) : \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{AlgSt})^{op} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}}$$

where St *is the* (2, 1)*-category of algebraic stacks with smooth topology.*
Morever, for any smooth atlas $x : X \to \mathcal{X}$ of an algebraic stack $\mathcal{X}$, one has

$$\mathcal{D}_{et}(\mathcal{X}, \mathbf{Q}_l) \cong \lim(\mathcal{D}_{et}(X, \mathbf{Q}_l) \rightleftarrows \mathcal{D}_{et}(X \times_{\mathcal{X}} X, \mathbf{Q}_l) \substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \cdots) \tag{3.13}$$

where the limit is over the Čech nerve of x*.*

Proof. We first construct the functor on the level of algebraic spaces.
Since $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}}$ admits small limits (Proposition A.18.15), we can apply Theorem 3.4.1 to the étale sheaf $\mathcal{D}_{et}(-\mathbf{Q}_l)$ and $\mathrm{St}_{\mathcal{C}} := \mathrm{Algsp}_{et-loc} = \mathrm{Algsp}$ (as every algebraic space admits a surjective étale covering by a scheme). Thus we get an ∞-sheaf $\mathcal{D}_{et}(-, \mathbf{Q}_l) : \mathrm{N}(\mathrm{Algsp})^{op} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}}$ on $(\mathrm{N}(\mathrm{Algsp}), et)$.
We now apply Theorem 3.4.1 again to the étale-sheaf $\mathcal{D}_{et}(-, \mathbf{Q}_l) : \mathrm{N}(\mathrm{Algsp})^{op} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}}$ and $\mathrm{St}_{\mathcal{C}} := \mathrm{Algst}_{et-loc} = \mathrm{AlgSt}$ to get an ∞-sheaf on $(\mathrm{AlgSt}, \mathrm{smooth})$.
The description of $\mathcal{D}_{et}(\mathcal{X}, \mathbf{Q}_l)$ as a limit is a direct consequence of Eq. (3.10).

□

Remark 3.5.2. The above definition agrees with the definition of Liu and Zheng by [LZ17, Proposition 5.3.5] which expresses their construction in terms of the Čech nerve of a covering as in Eq. (3.10).
The construction using ∞-categories has the advantage that the pullback functors f^* are built into the theory and thus avoid the technical problems with the lisse-étale topos (see [Ols05]).

3.5.2 The motivic stable homotopy category of an algebraic stack.

In [Rob15], Robalo explains that the construction of motivic stable homotopy theory ([MV99]) can be viewed as a functor taking values in ∞-categories.

Let Sch_{fd} denote the category of Noetherian schemes with finite Krull dimension. Robalo uses the construction of $\mathcal{SH}$ to define a functor

$$\mathcal{SH}^{\otimes}(-) : \mathrm{Sch}^{op}_{fd} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}}). \quad \text{[Rob14, Section 9.1]} \tag{3.14}$$

Let us unravel the information contained in this functor. As the classical SH admits a symmetric monoidal structure, the ∞-category $\mathcal{SH}^{\otimes}(S)$ is a symmetric monoidal ∞-category, i.e. it comes equipped with a coCartesian fibration $p_S : \mathcal{SH}^{\otimes}(S) \to \mathrm{N}(\mathrm{Fin}_*)$ (Definition B.1.1 and Definition A.8.10) such that $\mathcal{SH}^{\otimes}(S)_{\langle n \rangle} \cong \mathcal{SH}^{\otimes}(S)^{\times n}_{\langle 1 \rangle}$. Denote $\mathcal{SH}(S) := \mathcal{SH}^{\otimes}(S)_{\langle 1 \rangle}$. The coCartesian fibration encodes the symmetric monoidal structure of the ∞- category $\mathcal{SH}(S)$ in a coherent way.

The ∞-category $\mathcal{SH}(S)$ is also presentable as it arises from localization of presheaves of smooth schemes over S ([Lur09, Theorem 5.5.1.1]). It is also a stable ∞-category (Definition A.18.7) as it inherits the triangulated structure of SH constructed in [MV99]. As pullback morphism for a morphism f in Sch_{fd} is colimit preserving, the ∞-category $\mathcal{SH}^{\otimes}(S)$ lands in the ∞-category of presentable stable symmetric monoidal ∞-categories which is denoted by $\mathrm{CAlg}(\mathrm{Pr}^{L}_{\mathrm{stb}})$.
The ∞-category $\mathcal{SH}(S)$ is the called the stable motivic homotopy category of S.
As explained in [Rob14, Remark 9.3.1], the stable motivic homotopy theory can be extended to all schemes. The functor $\mathcal{SH}^{\otimes}(-)$ is a Nisnevich sheaf ([Hoy17, Proposition 6.24]).

Corollary 3.5.3. *The functor $\mathcal{SH}^{\otimes}(-)$ extends to an ∞-sheaf*

$$\mathcal{SH}^{\otimes}_{\mathrm{ext}}(-) : \mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt})^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{\mathrm{stb}})$$

.

Morever, for any algebraic stack $\mathcal{X} \in \mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt})$ that admits a schematic atlas $x : X \to \mathcal{X}$, one has

$$\mathcal{SH}^{\otimes}_{\mathrm{ext}}(\mathcal{X}) \cong \lim \Big(\mathcal{SH}^{\otimes}(X) \rightrightarrows \mathcal{SH}^{\otimes}(X \times_{\mathcal{X}} X) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots \Big) \tag{3.15}$$

where the limit is over the Čech nerve of x.

Proof. The proof is similar to the case of $\mathcal{D}_{et}(-, \mathbf{Q}_l)$. As $\mathrm{CAlg}(\mathrm{Pr}^{L}_{\mathrm{stb}})$ admit small limits (Theorem B.2.8), we can apply Theorem 3.4.1 to the functor $\mathcal{SH}^{\otimes}(-)$ with $\mathcal{T} = \mathrm{Nis}$ and $\mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}}) = \mathsf{N}(\mathrm{Algsp})$. This gives us an ∞-sheaf $\mathcal{SH}^{\otimes}_{\mathrm{Algsp}}(-) : \mathsf{N}(\mathrm{Algsp})^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{\mathrm{stb}})$.
Applying the theorem again to the ∞-sheaf $\mathcal{SH}^{\otimes}_{\mathrm{Algsp}}(-)$ with $\mathsf{N}(\mathcal{C}) = \mathsf{N}(\mathrm{Algsp})$, $\mathcal{T} = \mathrm{Nis}$ and $\mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}}) = \mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt})$, one gets an ∞-sheaf $\mathcal{SH}^{\otimes}_{\mathrm{ext}}(-)$.
The limit description is a consequence of Eq. (3.10) applied to the functor $\mathcal{SH}^{\otimes}_{\mathrm{ext}}(\mathcal{X})$. □

Notation 3.5.4. For any algebraic stack $\mathcal{X} \in \text{Nis-locSt}$, we shall denote the underlying presentable stable ∞-category of the symmetric monoidal ∞-category $\mathcal{SH}^{\otimes}_{\mathrm{ext}}(\mathcal{X})$ by $\mathcal{SH}_{\mathrm{ext}}(\mathcal{X})$. We shall call $\mathcal{SH}_{\mathrm{ext}}(\mathcal{X})$ to be the *stable motivic homotopy category of $\mathcal{X}$*.

Remark 3.5.5. Recall from Corollary 3.3.6 that for quotient stacks $[X/G]$, the atlas $X \to [X/G]$ does not admit Nisnevich-local sections and thus we need to replace it by $X \times^{G} \mathrm{GL}_n$ to compute $\mathcal{SH}^{\otimes}_{\mathrm{ext}}$.
In [Hoy17], Hoyois defines SH for global quotient stacks by tame reductive groups. His construction a priori may depend on choice of presentation of the quotient stack. Our construction has the advantage that it is independent of such a choice and it morever allows us to drop the tameness assumption.

The description of the ∞-sheaf $\mathcal{SH}^{\otimes}_{\mathrm{ext}}(-)$ gives us the following functors.

Notation 3.5.6. 1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism in $\mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt})$. We denote the *pullback functor* $\mathcal{SH}^{\otimes}_{\mathrm{ext}}(f) : \mathcal{SH}^{\otimes}_{\mathrm{ext}}(\mathcal{Y}) \to \mathcal{SH}^{\otimes}_{\mathrm{ext}}(\mathcal{X})$ by $f^{*\otimes}$. We shall also write $f^* : \mathcal{SH}_{\mathrm{ext}}(\mathcal{Y}) \to \mathcal{SH}_{\mathrm{ext}}(\mathcal{X})$ as the functor $f^{*\otimes}$ on the level of underlying ∞-categories. As

f^* is a colimit preserving functor, the adjoint functor theorem ([Lur09, Corollay 5.5.2.9] which we recalled in Theorem A.15.9) says there exists a right adjoint

$$f_* : \mathcal{SH}_{\text{ext}}(X) \to \mathcal{SH}_{\text{ext}}(Y)$$

which we call the *pushforward functor*.

2. As $\mathcal{SH}^{\otimes}_{\text{ext}}(\mathcal{X})$ is a symmetric monoidal ∞-category, we shall denote the functor induced by the symmetric monoidal structure by

$$- \otimes - : \mathcal{SH}_{\text{ext}}(\mathcal{X}) \times \mathcal{SH}_{\text{ext}}(\mathcal{X}) \to \mathcal{SH}_{\text{ext}}(\mathcal{X}).$$

For a scheme X, the ∞-category $\mathcal{SH}^{\otimes}(X)$ is closed. Let us explain this notion briefly. Given any two objects E and E' in $\mathcal{SH}(X)$, one has objects $\text{Hom}(E, E')$ and $\text{Hom}(E', E)$ in $\mathcal{SH}(X)$ with maps $\text{Hom}(E, E') \otimes E \to E'$ and $\text{Hom}(E', E) \otimes E' \to E$ satisfying usual universal properties. In other words, the tensor product realized as a functor $\mathcal{SH}(\mathcal{X}) \to \text{Fun}(\mathcal{SH}(X), \mathcal{SH}(X))$ factorizes via $\text{Fun}^{L}(\mathcal{SH}(X), \mathcal{SH}(X))$ ([Lur17, Definition 4.1.15]). We have the following proposition.

Proposition 3.5.7. *[LZ17, Remark 1.5.3] For any $\mathcal{X} \in$ Nis-locSt, the ∞-category $\mathcal{SH}^{\otimes}_{\text{ext}}(\mathcal{X})$ is closed.*

Remark 3.5.8. Let $x : X \to \mathcal{X}$ be an atlas admitting Nisnevich-local sections. Then we have a functor

$$p^{\otimes} : N(\Delta) \to \text{CAlg}(\text{Pr}^{L}_{\text{stb}})$$

induced by the Čech nerve of x. As $\mathcal{SH}^{\otimes}(X^{n}_{\mathcal{X}})$ is closed for every n. Then by [LZ17, Remark 1.5.3], we get that the limit of $p^{\otimes}$ i.e.. $\mathcal{SH}^{\otimes}_{\text{ext}}(\mathcal{X})$ is closed.

Notation 3.5.9. For any objects $\mathcal{E}, \mathcal{E}' \in \mathcal{SH}_{\text{ext}}(\mathcal{X})$, we shall denote $\text{Hom}_{\mathcal{SH}_{\text{ext}}(\mathcal{X})}(\mathcal{E}, \mathcal{E}')$ to be the internal Hom.

Thus we have defined four functors $f^*, f_*, - \otimes -$ and $\text{Hom}(-,-)$ along with the functor $\mathcal{SH}^{\otimes}_{\text{ext}}(-)$. In the next chapter we explain how to construct $f_!$ for representable morphism of algebraic stacks which are separated and finite type.

CHAPTER 4

ENHANCED OPERATIONS FOR STABLE HOMOTOPY THEORY OF ALGEBRAIC STACKS

In the previous chapter, we have extended the stable homotopy functor $\mathcal{SH}$ from schemes to algebraic stacks. We have also defined the four functors $f^*, f_*, - \otimes -$ and $\mathrm{Hom}(-,-)$. The goal of this chapter is to construct the functors $f_!$, $f^!$ and prove the base change and projection formula (Theorem 4.1.1).

The key idea is to construct these functors and proving the above mentioned properties via the enhanced operation map due to Liu and Zheng ([LZ17]). The enhanced operation map is a functor which encodes all of this information. As $\mathcal{SH}$ for schemes satisfy relations among six operations , the enhanced operation map can be constructed on the level of schemes (see [Rob14, Section 9.4]). We shall extend the enhanced operation map from schemes to algebraic stacks which shall prove the theorem.

Let us briefly outline the sections in the chapter. The first section states the theorem and motivates the notion of an enhanced operation map. The second section introduces the generalized notion of bisimplicial sets and marked simplicial sets which are convenient to encode simplicial versions of the base change square. The third section introduces the enhanced operation map of $\mathcal{SH}$ on the level of schemes and explain how it encodes all of the properties that we want. In the last section, we prove the theorem by extending the enhanced operation map to Nis-locSt. The proof of extension uses the same idea as the proof of Theorem 3.4.1.

4.1 Statement of the theorem and motivation for enhanced operation map.

The extraordinary pushforward $f_!$ and extraordinary pull-back functors $f^!$ are defined for morphisms of schemes that are separated and of finite type ([Rob14, Theorem 9.4.8]). We will denote by $\mathrm{Sch}'_{\mathsf{fd}} \subset \mathrm{Sch}_{\mathsf{fd}}$ the category of schemes in which morphisms are separated and of finite type. With this notation, the functors $f_!$ and $f^!$ can be assembled into functors

$$\mathcal{SH}_! : \mathsf{N}(\mathrm{Sch}'_{\mathsf{fd}}) \to \mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}} \quad , \quad \mathcal{SH}^! : \mathsf{N}(\mathrm{Sch}'_{\mathsf{fd}})^{\mathsf{op}} \to \mathrm{Pr}^{\mathsf{R}}_{\mathsf{stb}} \, .$$

We shall denote by Nis-locSt$'$ $\subset$ Nis-locSt the subcategory in which morphisms are representable and separated of finite type. Note that for a representable morphism, separated is equivalent to the fact that the diagonal is a closed immersion ([Sta21, Tag 04YS]).

Theorem 4.1.1. *The functors $\mathcal{SH}_!$ and $\mathcal{SH}^!$ extend to functors*

$$\mathcal{SH}_{\mathrm{ext}\,!} : \mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt}') \to \mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}$$

and

$$\mathcal{SH}^!_{\mathrm{ext}} : \mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt}')^{\mathsf{op}} \to \mathrm{Pr}^{\mathsf{R}}_{\mathsf{stb}} \, .$$

These functors satisfy:

1. ***(Base change)*** *Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array} \tag{4.1}$$

be a pullback diagram in Nis-locSt *where* f *and* f' *are separated of finite type. Then the diagram*

$$\begin{array}{ccc} \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}) & \xrightarrow{g'^*} & \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}') \\ \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f'_!} \\ \mathcal{SH}_{\mathrm{ext}}(\mathcal{Y}) & \xrightarrow{g^*} & \mathcal{SH}_{\mathrm{ext}}(\mathcal{Y}') \end{array} \tag{4.2}$$

commutes in $\mathrm{CAlg}(\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}})$. *In other words we have an equivalence of functors*

$$\mathrm{Ex}(\Delta^*_{\#}) : g^* \circ f_! \cong f'_! \circ g'^*$$

in the functor category $\mathrm{Fun}(\mathcal{SH}_{\mathrm{ext}}(\mathcal{X}), \mathcal{SH}_{\mathrm{ext}}(\mathcal{Y}'))$.

2. ***(Projection formula)*** *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a morphism in* Nis-locSt$'$. *Given* $E \in \mathcal{SH}(\mathcal{X})$ *and* $E' \in \mathcal{SH}(\mathcal{Y})$, *there exists an equivalence*

$$f_!(E \otimes f^*(E')) \cong f_!(E) \otimes E'. \tag{4.3}$$

Remark 4.1.2. To prove the theorem, it suffices to construct the functor $\mathcal{SH}_{\mathrm{ext}\,!}$. Given $\mathcal{SH}_{\mathrm{ext}\,!}$, the functor $\mathcal{SH}^!_{\mathrm{ext}}$ can be defined by $\mathcal{SH}^!_{\mathrm{ext}} = (\mathcal{SH}_{\mathrm{ext}\,!})^{\mathsf{op}} : \mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt})^{\mathsf{op}} \to (\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}})^{\mathsf{op}} \cong \mathrm{Pr}^{\mathsf{R}}_{\mathsf{stb}}$. The equivalence $(\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}})^{\mathsf{op}} \cong \mathrm{Pr}^{\mathsf{R}}_{\mathsf{stb}}$ follows from [Lur09, Corollary 5.5.3.4].

We shall prove this theorem in Section 4.4. The proof of the theorem relies on extending a special kind of map from schemes to algebraic stacks. We call this map the *"enhanced operation map"* due to Liu and Zheng ([LZ17]). The existence of the enhanced operation map shall give us the lower shriek functors, base change formula and projection formula. We now try to motivate the source and target of this map.

Let us first motivate the target. This comes from the projection formula. The projection formula can viewed as a property in the ∞-category of module objects $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}})$. Let us briefly recall the ∞-categorical analog of classical module categories i.e. $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}})$.

The objects of $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}})$ are pairs $(\mathcal{C}^{\otimes}, \mathcal{M})$ where $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category (i.e. an object in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}})$) and $\mathcal{M}$ is a presentable stable ∞-category with a morphism $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ which incorporates the module structure of the object $\mathcal{M}$ (here $\mathcal{C}$ is the underyling ∞-category of $\mathcal{C}^{\otimes}$). A morphism $(\mathcal{C}^{\otimes}, \mathcal{M}) \to (\mathcal{C}'^{\otimes}, \mathcal{M}')$ in $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}})$ consists of morphisms of commutative algebra objects $u : \mathcal{C}^{\otimes} \to \mathcal{C}'^{\otimes}$ and a morphism $v : \mathcal{M} \to \mathcal{M}'$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}}$ which is $\mathcal{C}$-linear where $\mathcal{M}'$ is endowed with a $\mathcal{C}^{\otimes}$-module structure via u. In particular for objects $c \in \mathcal{C}$ and $m \in M$, one has an equivalence

$$v(c \otimes m) \cong u(c) \otimes v(m). \tag{4.4}$$

We recall the precise definition of module objects in Definition B.4.3.

In the context of stable homotopy theory of schemes, a morphism $f : X \to Y$ induces a monoidal pullback functor $f^{*\otimes} : \mathcal{SH}^{\otimes}(Y) \to \mathcal{SH}^{\otimes}(X)$. Then the pair $(\mathcal{SH}^{\otimes}(Y), \mathcal{SH}(X))$ is an example of a module object. We can visualize this as an ∞-categorical generalization of the statement that a morphism $g : A \to B$ of rings makes B an A-module. Also for any object $Z \in \mathrm{Sch}_{\mathrm{fd}}$, the pair $(\mathcal{SH}^{\otimes}(Z), \mathcal{SH}(Z))$ is an object in $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}})$ (the module structure is induced by the tensor product).

The projection formula is equivalent to the statement that the pair of morphisms

$$(\mathrm{id}, f_!) : (\mathcal{SH}^{\otimes}(Y), \mathcal{SH}(X)) \to (\mathcal{SH}^{\otimes}(Y), \mathcal{SH}(Y)) \tag{4.5}$$

is a morphism in $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}})$. This follows from the condition of module morphism (Eq. (4.4) applied to $v = f_!$ and $c = \mathrm{id}$) in this context. The above discussion motivates that $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}})$ will be the target of the enhanced operation map.

Let us motivate the source of the enhanced operation map. As stated before, the map encodes both the lower shriek functors and the base change formula. To combine these, one defines a simplicial set where the 1-simplices are cartesian squares

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array} \tag{4.6}$$

where f (and thus f') is representable, separated and finite type.

Thus the enhanced operation map is a functor from a simplicial set whose simplices consists of pullback squares as above and takes values in $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}})$.

The source of the enhanced operation map naturally defines a bisimplicial set in which vertical arrows are separated and finite type, which is called a bi-marked simplicial set ([LZ12, Definition 3.9]). As the functor $f_!$ is usually constructed by combining constructions for open embeddings and proper morphisms, it will be useful to have a more general notion of a multisimplicial set in which the class of arrows in some directions are restricted. Let us therefore recall these notions from the article of Liu and Zheng ([LZ12, Section 3]).

4.2 Multisimplicial, multi-marked and multi-tiled simplicial sets.

4.2.1 Multisimplicial sets.

Let I be a finite set and consider it as a discrete category.

Definition 4.2.1. [LZ12, Definition 3.1] An *I-simplicial set* is a functor:

$$\mathrm{Fun}(I,\Delta)^{op} = (\underbrace{\Delta \times \Delta \cdots \Delta}_{\text{I-times}})^{op} \to \mathrm{Sets}$$

We denote the category of I-simplicial sets by $\mathrm{Sets}_{I\Delta}$. If $I = \{1, 2, \cdots, k\}$, then we denote it by $\mathrm{Set}_{k\Delta}$.

Remark 4.2.2. By definition, $\mathrm{Sets}_{1\Delta} = \mathrm{Sets}_{\Delta}$ and similarly $\mathrm{Sets}_{2\Delta}$ is the category of bisimplicial sets.

Notation 4.2.3. We shall denote any object $(n_i)_{i\in I}$ of $\mathrm{Fun}(I,\Delta)$ by $\underline{n}$ We denote $\Delta^{\underline{n}}$ to be the I-simplicial set represented by $\prod_i \Delta^{n_i}$. For an I-simplicial set, we denote $S_{\underline{n}}$ by $S(\underline{n})$.

We discuss adjunctions between $\mathrm{Sets}_{k\Delta}$ and Sets_{Δ}.

Notation 4.2.4. [LZ12, Definition 3.3]

1. Denote $f : I = \{1, 2, \cdots, k\} \to \{1\}$ be the projection map. This induces the functor $\Delta \to \mathrm{Fun}(I,\Delta)$ which induces the *diagonal functor*:

$$\delta^*_k : \mathrm{Sets}_{k\Delta} \to \mathrm{Sets}_{\Delta}$$

which takes an k-simplicial set S to $\delta^*_k(S)$ which evaluated on $[n]$ is $S([n], [n], \cdots [n])$. This functor has a right adjoint:

$$\delta^k_* : \mathrm{Sets}_{\Delta} \to \mathrm{Sets}_{k\Delta}$$

which evaluated on S, defines a k-simplicial set defined as

$$\delta^k_*(S)_{\underline{n}} = \mathrm{Hom}_{\mathrm{Sets}_{\Delta}}(\prod_{i\in I} \Delta^{n_i}, S)$$

2. Similarly an injection of sets $f : J \hookrightarrow I$ induces a functor $(\Delta_f)^* : \mathrm{Sets}_{J\Delta} \to \mathrm{Sets}_{I\Delta}$ induced from f. It has a right adjoint, which we denote by
$$\epsilon^I_J : \mathrm{Sets}_{I\Delta} \to \mathrm{Sets}_{J\Delta}$$
defined by
$$\epsilon^I_J(S)(\underline{n}) = S((\underline{n}, 0))$$
where we write $(\underline{n}, 0)$ for the vector with entries 0 for $i \neq j$. We call the map ϵ^I_J as *restriction functor.*
If $I = \{1, 2, \cdots, k\}$ and $J = \{j\}$, then we denote it by ϵ^k_j.

3. Given $I = \{1, 2, \cdots, k\}$ and $J \subset I$. We have the *partial opposite* functor
$$\mathrm{op}^I_J : \mathrm{Sets}_{k\Delta} \to \mathrm{Sets}_{k\Delta}$$
defined by taking opposite edges along the directions $j \in J$. Using this notion, we define the *twisted diagonal* functor as
$$\delta^*_{k,J} := \delta^*_I \circ \mathrm{op}^I_J : \mathrm{Sets}_{k\Delta} \to \mathrm{Sets}_\Delta\,.$$

Example 4.2.5. 1. The map δ^2_* takes a simplicial set S to the bisimplicial set $\delta^2_* S$ whose (n_1, n_2) simplices are $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^{n_1} \times \Delta^{n_2}, S)$. If $S = N(C)$ where C is an ordinary category, then these are just $n_1 \times n_2$ grids in $\mathcal{C}$.
The map δ^*_2 takes a bisimplicial set to its diagonal simplicial set. For $S = \delta^2_* N(C)$, the n-simplices of the simplicial set $\delta^*_2(\delta^2_* N(C))$ are morphisms $\Delta^n \times \Delta^n \to N(C)$ (in other words these are $n \times n$ grids in C).

2. The maps ϵ^2_1 and ϵ^2_2 send a bisimplicial set $S' : (\Delta \times \Delta)^{op} \to \mathrm{Sets}$ to the simpicial sets $S'|_{\Delta \times [0]}$ and $S'|_{[0] \times \Delta}$ respectively, i.e.. these are the restrictions to the first row and column of the bisimplicial set.

3. For $k = 1$, the twisted diagonal functor sends a simplicial set S to S^{op}.For $k = 2$, the partial opposite functor $\mathrm{op}^2_{\{1\}}$ takes a bisimplicial set S and sends to the bisimplicial set S' which when restricted to direction 1 gives the simplicial set $(\epsilon^2_1 S)^{op}$ and when restricted to direction 2 gives the simplicial set $\epsilon^2_2 S$. In order to understand it more clearly, let us consider the bisimplicial set $\delta^2_* N(C)$. Then the n simplices of the simplicial se $\delta^*_{2,\{1\}}(\delta^2_* N(C))$ are given by $n \times n$ grids $(\Delta^n)^{op} \times \Delta^n \to N(C)$.

4.2.2 Multi-marked and multi-tiled simplicial sets.

Definition 4.2.6. [LZ12, Definition 3.9] An I-*marked simplicial set* is the data $(S, \mathcal{E} := \{\mathcal{E}_i\}_{i \in I})$ where S is a simplicial set and $\mathcal{E}$ is a set of edges $\mathcal{E}_i$ containing every degenerate edge of S. A morphism between I-marked simplicial sets $(S, \mathcal{E})$ and $(S', \mathcal{E}')$ is a morphism of simplicial sets $f : X \to X'$ with the property $f(\mathcal{E}_i) \subset \mathcal{E}'_i$. We denote the category of I-marked simplicial sets as $\mathrm{Sets}^{I+}_\Delta$. If $I = \{1, 2, \cdots, k\}$, we denote the category of I-marked simplicial sets by $\mathrm{Sets}^{k+}_\Delta$.

Remark 4.2.7. An I-marked simplicial set is said to be an I-marked ∞-category if the underlying simplicial set is an ∞-category.
For $k = 1$, we get the notion of marked simplicial sets defined in [Lur17, Section 3.1].

Notation 4.2.8.

1. Given any I-simplicial set S, we can define an I-marked simplicial set $\delta^*_{I+}(S') = (\delta^*_I S', \mathcal{E} = \{(\epsilon^I_i S)_1\}_{i\in I})$. When $k = 2$, the marked simplicial set $\delta^*_{+2}(S')$ consists of the diagonal simplicial set of S' with the marked edges being the edges of the simplicial set of the first row and first column of the bisimplicial set.

2. Given any I-marked simplicial set $(S, \mathcal{E})$, we can define an I-simplicial set $\delta^{I+}_*(S, \mathcal{E})$ as the sub I-simplicial set of $\delta^I_* S$ which consists only of edges $\mathcal{E}_i$ in simplicial set $\epsilon^I_i(S)$.

This notion yields us to define the notion of restricted simplicial nerve.

Definition 4.2.9. [LZ12, Definition 3.10] Let $(S, \mathcal{E})$ be an I-marked simplicial set, then we define the *restricted* I-*simplicial nerve* as

$$S_{\mathcal{E}} := \delta^{I+}_*(S, \mathcal{E})$$

Example 4.2.10. Let $(S, \mathcal{E}) = (N(\mathrm{Sch}), \{P, O\})$ where P and O are the set of proper morphisms and open immersions respectively. Then $S_{\mathcal{E}}$ is the bisimplicial subset of the bisimplicial set $\delta^2_* N(\mathrm{Sch})$ which consists of only proper morphisms as edges in the simplicial set $\epsilon^2_1(\delta^2_* N(\mathrm{Sch}))$ and open immersions as edges in the simplicial set $\epsilon^2_2(\delta^2_* N(\mathrm{Sch}))$.

Definition 4.2.11. [LZ12, Definition 3.12] An *I-tiled simplicial set* is the data $(S, \mathcal{E} = \{\mathcal{E}_i\}_{i\in I}, \mathbf{Q} = \{\mathbf{Q}_{ij}\}_{i,j\in I, i\neq j})$ where $(X, \mathcal{E})$ is a marked simplicial set and $\mathbf{Q}$ is a collection of set of squares $\mathbf{Q}_{ij}$ (i.e. $\Delta^1 \times \Delta^1 \to S$) such that

1. the set of squares $\mathbf{Q}_{ij}$ and $\mathbf{Q}_{ji}$ are obtained from each other by transposition.

2. The vertical arrows of each square in $\mathbf{Q}_{ij}$ are in $\mathcal{E}_i$ and the horizontal arrows are in $\mathcal{E}_j$.

3. To every edge in $\mathcal{E}_i$, there is a square in $\mathbf{Q}_{ij}$ induced by the map $\mathrm{id} \times s^0_0$.

A morphism of I-tiled simplicial sets $f : (S, \mathcal{E}, \mathbf{Q}) \to (S', \mathcal{E}', \mathbf{Q}')$ which maps $f(\mathcal{E}_i) \subset \mathcal{E}'_i$ and $f(\mathbf{Q}_{ij}) \subset \mathbf{Q}'_{ij}$. We denote the category of I-tiled simplicial sets by $\mathrm{Sets}^{I\square}_{\Delta}$.

Notation 4.2.12. [LZ12, Remark 3.13]

1. Given any I-simplicial set S, we define an I-tiled simplicial set $\delta^*_{I\square}(S) := (\delta^*_I S, \mathcal{E}, \mathbf{Q})$ where $\mathcal{E} = \{\mathcal{E}_i = (\epsilon^I_i S')_1\}_{i\in I}$ and $\mathbf{Q} = \{\mathbf{Q}_{ij} = \mathrm{Hom}(\Delta^1 \times \Delta^1, \delta^*_2 \epsilon^I_{i,j}(S))\}_{i,j\in I, i\neq j}$.

2. Given any I-tiled simplicial set $(S', \mathcal{E}', \mathbf{Q}')$, we define an I-simplicial set $\delta^{I\square}_*((S', \mathcal{E}', \mathbf{Q}'))$ as the I-simplicial subset of $\delta^{I+}_*(S', \mathcal{E}')$ such that for $j, k \in I$ and $j \neq k$, every square in the bisimplicial set $\epsilon^I_{jk}(\delta^{I+}_*(S', \mathcal{E}'))$ lies in $\mathbf{Q}_{jk}$.

Remark 4.2.13. In the above notation, let us explain what does a square in a bisimplicial set means. Let S be a bisimplicial set. Given any $(1,1)$-simplex of S, we can define a square in the diagonal simplicial set $\delta^*_2 S$ as follows. A $(1,1)$-simplex corresponds to a morphism $\tau : \Delta^{(1,1)} \to S$. Applying the functor $\delta^*_2(-)$, we get a morphism

$$\delta^*_2(\tau) : \Delta^1 \times \Delta^1 \to \delta^*_2 S.$$

If $S = N(\mathrm{Sch}')_{P,O}$, then a square in S corresponds to a morphism $\Delta^1 \times \Delta^1 \to N(\mathrm{Sch}')$ where horizontal arrows are proper and vertical arrows are open.

Definition 4.2.14. [LZ12, Definition 3.16] Let $\mathcal{C}$ be and ∞-category and $\mathcal{E}_1, \mathcal{E}_2$ be set of edges, denote $\mathcal{E}_1 \star^{\mathrm{cart}} \mathcal{E}_2$ be the set of Cartesian squares. For an I-marked ∞-category $(\mathcal{C}, \mathcal{E} := \{\mathcal{E}_i\}_{i\in I})$, we denote $\mathcal{E}^c_{ij} := \mathcal{E}_i \star^{\mathrm{cart}} \mathcal{E}_j$. Denote $(\mathcal{C}, \mathcal{E}, \mathcal{E}^c)$ to be the I-tiled ∞-category. We define the *Cartesian* I-*simplicial nerve* to be the I-simplicial set

$$\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}} := \delta^{I\square}_{*}((\mathcal{C}, \mathcal{E}, \mathcal{E}^c))$$

Example 4.2.15. The bisimplicial set $N(\mathrm{Sch})^{\mathrm{cart}}_{P,O}$ is the sub-bisimplicial set of $\delta^2_* N(\mathrm{Sch})$ which consists of proper morphisms as edges in one direction, open immersions as edges in other and every square formed by open and proper morphisms is a pullback square.

Let us understand the simplicial set $\delta^*_k N(C)^{\mathrm{cart}}_{\mathcal{E}}$ which will be the source of the enhanced operation map. A n-simplex of $\delta^*_k N(C)^{\mathrm{cart}}_{\mathcal{E}}$ is a morphism $\sigma_n : \underbrace{\Delta^n \times \Delta^n \cdots \times \Delta^n}_{\text{k-times}} \to N(C)$ such that every edge $\sigma_n|_i : \Delta^1 \to N(C)$ in direction i lies in $\mathcal{E}_i$ for every $i \in I$ and for every $j \neq j' \in I$, the square $\sigma_n|_{j,j'} : \Delta^1 \times \Delta^1 \to N(C)$ is a pullback square formed by edges $\mathcal{E}_j$ and $\mathcal{E}_{j'}$. In case $k = 2$ and $(C, \mathcal{E}) = (\mathrm{Sch}, \{P, O\})$, the n-simplices of $\delta^*_2 N(\mathrm{Sch}')^{\mathrm{cart}}_{P,O}$ are $n \times n$ grids of the form

$$\begin{array}{ccccccc}
X_{00} & \longrightarrow & X_{01} & \longrightarrow & \cdots & & X_{0n} \\
\downarrow & & \downarrow & & & & \downarrow \\
X_{10} & \longrightarrow & X_{11} & \longrightarrow & \cdots & & X_{1n} \\
 & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & \\
X_{n1} & \longrightarrow & X_{n2} & & \cdots & & X_{nn}
\end{array} \tag{4.7}$$

where vertical arrows are open, horizontal arrows are proper and each square is a pullback square.

Notation 4.2.16. Let $(N(C), \{\mathcal{E}_1, \mathcal{E}_2\})$ be a marked ∞-category. For any $i = 1$ or 2, let $N(C_{\mathcal{E}_i})$ be the subcategory of $N(C)$ spanned by edges in $\mathcal{E}_i$. Then we have a natural map

$$\mathrm{dir}_{\mathcal{E}_i} : N(C_{\mathcal{E}_i}) \to \delta^*_2 N(C)^{\mathrm{cart}}_{\mathcal{E}}$$

which on the level of 1-simplices sends a morphism $f : X \to Y$ to a square of the form

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \xrightarrow{f} & Y.
\end{array} \tag{4.8}$$

We call the map $\mathrm{dir}_{\mathcal{E}_i}$ as the *restriction map along direction* i.

4.3 The enhanced operation map for $\mathcal{SH}(X)$.

In this section, we introduce the enhanced operation map on schemes. Let us first introduce the formal setup of the enhanced operation map as mentioned in [Rob14, Section 9.4].

4.3.1 Setup of the enhanced operation map.

Notation 4.3.1. Let $\mathrm{Sch}_{\mathsf{fd}}$ be the category of Noetherian schemes of finite Krull dimension Let

$$\mathcal{D}^{\otimes} : \mathsf{N}(\mathrm{Sch}_{\mathsf{fd}})^{\mathsf{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}})$$

be a functor. The underlying ∞-category of the symmetric monoidal ∞-category $\mathcal{D}^{\otimes}(X)$ is denoted by $\mathcal{D}(X)$. For a morphism of schemes $f : X \to Y$, we shall denote the pullback functor $\mathcal{D}(f) : \mathcal{D}(Y) \to \mathcal{D}(X)$ by f^*. It is a colimit preserving functor. Thus by adjoint functor theorem, there exists a right adjoint f_*. We assume the functor $\mathcal{D}$ has the following properties:

1. For any smooth morphism of finite type f, f^* has a left adjoint $f_\#$ such that:

   (a) (Smooth projection formula) For any $E \in \mathcal{D}(Y)$ and $B \in \mathcal{D}(X)$, the natural map formed by adjunction
   $$f_\#(E \otimes f^*(B)) \to f_\# E \otimes B \tag{4.9}$$
   is an equivalence.

   (b) (Smooth base change) For a cartesian square of schemes
   $$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \tag{4.10}$$
   with f smooth of finite type, the commutative square
   $$\begin{array}{ccc} \mathcal{D}(X') & \xleftarrow[\{f'\}^*]{} & \mathcal{D}(Y') \\ {\scriptstyle \{g'\}^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ \mathcal{D}(X) & \xleftarrow[f^*]{} & \mathcal{D}(Y) \end{array} \tag{4.11}$$
   is horizontally left-adjointable ([Lur17, Definition 4.7.4.13] which we recall in Definition A.13.3), i.e.. there exists a commutative square
   $$\begin{array}{ccc} \mathcal{D}(X') & \xrightarrow{\{f'\}_\#} & \mathcal{D}(Y') \\ {\scriptstyle \{g'\}^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ \mathcal{D}(X) & \xrightarrow{f_\#} & \mathcal{D}(Y) \end{array} \tag{4.12}$$

2. For $f : Y \to X$ a proper morphism of schemes, f^* admits a right adjoint functor f_* with the following properties:

(a) (Proper projection formula) For $E \in \mathcal{D}(Y)$ and $B \in \mathcal{D}(X)$, the natural map

$$f_*(E) \otimes B \to f_*(E \otimes f^*(B)) \tag{4.13}$$

is an equivalence.

(b) (Proper base change) For the cartesian square in Eq. (4.10), the induced pullback square Eq. (4.11) is horizontally right adjointable. In other words, the square commutes

$$\begin{array}{ccc} \mathcal{D}(X') & \xrightarrow{f'_*} & \mathcal{D}(Y') \\ {\scriptstyle \{g'\}^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ \mathcal{D}(X) & \xrightarrow{f_*} & \mathcal{D}(Y) \end{array} \tag{4.14}$$

3. (Support property) For a cartesian diagram of schemes in Eq. (4.10) where f is an open immersion and g is an proper,the commutative diagram in Eq. (4.12) written as square

$$\begin{array}{ccc} \mathcal{D}(X) & \xrightarrow{f_\#} & \mathcal{D}(Y) \\ \downarrow{\scriptstyle \{g'\}^*} & & \downarrow{\scriptstyle g^*} \\ \mathcal{D}(X') & \xrightarrow{\{f'\}_\#} & \mathcal{D}(Y') \end{array} \tag{4.15}$$

is horizontally right adjointable, i.e. the square

$$\begin{array}{ccc} \mathcal{D}(X') & \xrightarrow{\{f'\}_\#} & \mathcal{D}(Y') \\ \downarrow{\scriptstyle g'_*} & & \downarrow{\scriptstyle g_*} \\ \mathcal{D}(X) & \xrightarrow{f_\#} & \mathcal{D}(Y) \end{array} \tag{4.16}$$

commutes.

Example 4.3.2. The functor $\mathcal{SH}^{\otimes}$ satisfies the conditions of Notation 4.3.1 ([Rob14, Example 9.4.6]).

Notation 4.3.3. Let

$$\begin{array}{ccc} Y_0 & \xrightarrow{u} & Y_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\ X_0 & \xrightarrow{v} & X_1 \end{array} \tag{4.17}$$

be an edge in $\mathrm{Fun}(\Delta^1, \mathrm{Sch}_{fd})$. We want to denote some specific collection of edges in $\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathrm{Sch}_{fd}))$ as follows:

1. $F :=$ all such squares such that u and v are separated morphisms of finite type.

2. $\mathrm{ALL} :=$ all edges in $\mathrm{Fun}(\Delta^1, \mathrm{N}(\mathrm{Sch}_{fd}))$.

Theorem 4.3.4. *[LZ17, Section 3.2] Given a functor $\mathcal{D}^{\otimes} : \mathrm{N}(\mathrm{Sch}_{fd})^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{stb})$ which satisfies properties in Notation 4.3.1, then there exists an enhanced operation map*

$$\mathrm{EO}(\mathcal{D}^{\otimes}) : \delta^*_{2,\{2\}} \mathrm{Fun}(\Delta^1, \mathrm{N}(\mathrm{Sch}_{fd}))^{cart}_{F,\mathrm{ALL}} \to \mathrm{Mod}(\mathrm{Pr}^{L}_{stb}). \tag{4.18}$$

which when restricted to direction F*, gives us the lower shriek functor* $\mathcal{SH}_!$*. Morever, we get the projection and base change formulas by evaluating at specific simplicies of* $\delta^*_{2,\{2\}}\mathrm{Fun}(\Delta^1, N(\mathrm{Sch}_{fd}))^{cart}_{F,ALL}$.

Remark 4.3.5. 1. The above theorem can be applied to the functor $\mathcal{D}^{\otimes} = \mathcal{SH}^{\otimes}$ as $\mathcal{SH}^{\otimes}$ satisfies all conditions of Notation 4.3.1. This constructs the lower shriek functors, projection formula and base change (see [Rob14, Theorem 9.4.36]).

2. The enhanced operation map takes values in $\mathrm{Mod}(\mathrm{Pr}^L_{stb})$ as we wanted. Let us try to explain the source of enhanced operation and justify the motivation that we gave in the beginning of this chapter. In order to encode the module objects, the source consists morphisms of schemes as objects. Hence this motivates considering the functor category $\mathrm{Fun}(\Delta^1, \mathrm{Sch})$. The functor $\delta^*_{2,\{2\}}(-)$ encodes pullback squares as 1-simplices as we explained. Taking opposite direction along $\{2\}$ is motivated from the fact that the pullback functor is contravariant.

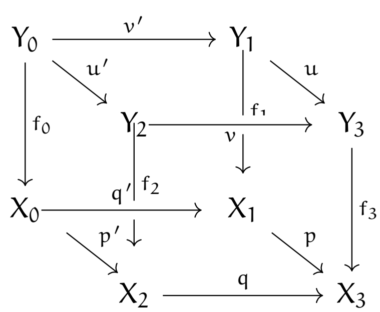
3. The construction of the enhanced operation map is technical and is recalled in Appendix D.3. It involves the theorem of partial adjoints ([LZ17, Proposition 1.4.4]) and ∞-categorical gluing for compactifiable morphisms ([LZ12, Theorem 0.1]). We recall both of the theorems and give a brief idea of proof of gluing in Appendix D.1. In the next section we explain how the enhanced operation map is defined on 0 and 1-simplices. The description of $\mathrm{EO}(\mathcal{D}^{\otimes})$ on lower simplices shall help us to understand how the lower shriek functors, projection formula and base change are encoded in $\mathrm{EO}(\mathcal{D}^{\otimes})$.

4.3.2 Understanding the map $\mathrm{EO}(\mathcal{D}^{\otimes})$.

Let us explain the map $\mathrm{EO}(\mathcal{D}^{\otimes})$ on the level of 0 and 1 simplices. The 0 and 1 simplices of $\delta^*_{2,\{2\}}(\mathrm{Fun}(\Delta^1, N(\mathrm{Sch}_{fd})))_{F,ALL}$ are:

1. 0- simplices are maps of schemes $f : Y \to X$.

2. A morphism from $f_0 : Y_0 \to X_0$ to $f_3 : Y_3 \to X_3$ is a morphism of the form $\Delta^1 \times \Delta^1 \to \mathrm{Fun}(\Delta^1, N(\mathrm{Sch}_{fd}))$ with conditions of edges and pullback squares. Explicitly, it is a cube of the form



(4.19)

where the diagonal maps are separated morphisms of finite type and the top and bottom squares are cartesian. The horizontal squares are considered on the opposite direction.

In general, n-simplices of the source of $\mathrm{EO}(\mathcal{D}^{\otimes})$ are maps

$$F_n : \Delta^n \times (\Delta^n)^{op} \times \Delta^1 \to \mathrm{Sch}_{fd}$$

such that $F_n|_{\Delta^n \times (\Delta^n)^{op} \times \{i\}}$ for $i = 0, 1$ determines an n-simplex in $\delta^*_{2,\{2\}}(\mathrm{Sch}_{fd})^{cart}_{F',ALL'}$ where F' is set of edges which are separated and finite type and ALL' are all morphisms of schemes.

Let us explain what $EO(\mathcal{D}^{\otimes})$ does on the level of 0 and 1 simplices.

1. $EO(\mathcal{D}^{\otimes})(f) := (\mathcal{D}^{\otimes}(X), \mathcal{D}(Y))$ where $f : Y \to X$ is an arbitrary morphism of schemes. Here $\mathcal{D}(Y)$ is a $\mathcal{D}^{\otimes}(X)$-module via f^*.
2. For a cube of the form Item 2, $EO(\mathcal{D}^{\otimes})$ sends the cube to a morphism of modules
$$(\mathcal{D}(X_0)^{\otimes}, \mathcal{D}(Y_0)) \to (\mathcal{D}(X_3)^{\otimes}, \mathcal{D}(Y_3)).$$
This shall help us to encode the projection formula which is a statement formulated as morphism of modules.

4.3.3 Extraordinary pushforward, base change and projection formula.

Now we explain how the extraordinary pushforward, projection formula and base change are encoded in the map $EO(\mathcal{D}^{\otimes})$.

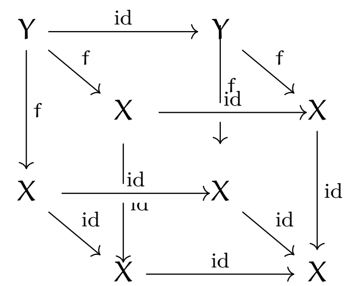
1. **The enhanced pullback:** The map $EO(\mathcal{D}^{\otimes})$ encodes the map $\mathcal{D}^{\otimes}$. When restricted the map along the direction ALL (Notation 4.2.16), then the induced map
$$EO(\mathcal{D}^{\otimes})^* : \mathrm{Sch}^{op}_{fd} \xrightarrow{X \to (X \to \mathrm{Spec}\,\mathbf{Z})} \mathrm{Fun}(\Delta^1, \mathrm{Sch}_{fd})^{op} \xrightarrow{EO(\mathcal{D}^{\otimes}) \circ \mathrm{dir}_{ALL}} \mathrm{Mod}(\mathrm{Pr}^L_{stb})$$
sends a scheme X to the module object $(\mathcal{D}^{\otimes}(X), \mathcal{D}(X))$ and sends a morphism of schemes $f : X \to Y$ to the pullback morphism
$$(\mathrm{id}, f^*) : (\mathcal{D}^{\otimes}(\mathrm{Spec}\,\mathbf{Z}), \mathcal{D}(Y)) \to (\mathcal{D}^{\otimes}(\mathrm{Spec}\,\mathbf{Z}), \mathcal{D}(X)).$$
This is called the enhanced pullback map. Restricting it to the first coordinate, we get the map
$$\mathcal{D} : \mathrm{Sch}^{op}_{fd} \to \mathrm{Pr}^L_{stb}.$$
2. **The extraordinary pushforward:** We have a canonical map
$$\mathrm{dir}_F : \mathrm{Fun}(\Delta^1, \mathrm{Sch}'_{fd}) \to \delta^*_{2,\{2\}}(\mathrm{Fun}(\Delta^1, \mathrm{Sch}_{fd}))^{cart}_{F,ALL}$$
which is the restriction direction along F (Notation 4.2.16).
This induces the map
$$EO(\mathcal{D}^{\otimes})_! : \mathrm{Sch}'_{fd} \to \mathrm{Fun}(\Delta^1, \mathrm{Sch}'_{fd}) \xrightarrow{EO(\mathcal{D}^{\otimes}) \circ \mathrm{dir}_F} \mathrm{Mod}(\mathrm{Pr}^L_{stb})$$
where the first map is induced by $X \to (X \to \mathrm{Spec}\,\mathbf{Z})$.
Combining with the description of $EO(\mathcal{D}^{\otimes})$ in Section 4.3.2, the map $EO(\mathcal{D}^{\otimes})_!$ sends a morphism $g : Y \to X$ in Sch'_{fd} to a morphism of modules
$$(\mathrm{id}, g_!) : (\mathcal{D}^{\otimes}(\mathrm{Spec}\,\mathbf{Z}), \mathcal{D}(Y)) \to (\mathcal{D}^{\otimes}(\mathrm{Spec}\,\mathbf{Z}), \mathcal{D}(X)).$$

We call $\mathrm{EO}(\mathcal{D}^{\otimes})_!$ the *enhanced extraordinary pushforward map*.
Via the restriction functor $\mathrm{Mod}(\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}})$, we get the extraordinary pushforward functor

$$\mathcal{SH}_! : \mathrm{Sch}'_{\mathsf{fd}} \to \mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}$$

which sends a morphism $f : X \to Y$ to the functor $f_! : \mathcal{SH}(X) \to \mathcal{SH}(Y)$.

3. **Projection formula:** For $f : Y \to X$ be a separated and finite type morphism, consider the cube:



(4.20)

Evaluating $\mathrm{EO}(\mathcal{D}^{\otimes})$ (using the description in Section 4.3.2) on this cube yields a morphism of modules

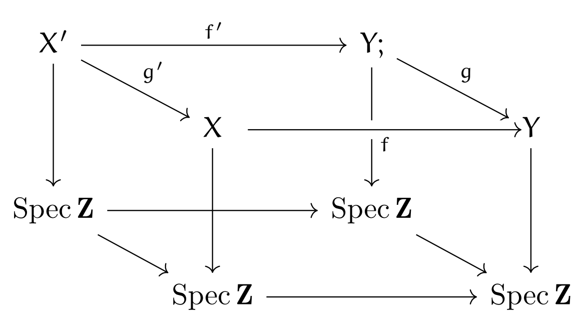
$$(\mathrm{id}, f_!) : (\mathcal{D}^{\otimes}(X), \mathcal{D}(Y)) \to (\mathcal{D}^{\otimes}(X), \mathcal{D}(X))$$

where $\mathcal{D}(Y)$ is a $\mathcal{D}(X)^{\otimes}$-module via f^* and $\mathcal{D}(X)$ is a $\mathcal{D}(X)^{\otimes}$-module via the tensor product. This equivalent to the module homomorphism (Eq. (4.4)) explained in the beginning of the chapter and hence it gives us the projection formula.

4. **Base change:** Consider the cartesian square of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \tag{4.21}$$

where f and f' are separated morphism of finite type. Let us explain how the map $\mathrm{EO}(\mathcal{D}^{\otimes})$ encodes the base change.
The above pullback square gives us a 1-simplex γ in $\delta^*_{2,\{2\}} \mathrm{Fun}(\Delta^1, \mathrm{Sch}_{\mathsf{fd}})^{\mathrm{cart}}_{\mathrm{F,ALL}}$ which is cube:



(4.22)

The above 1-simplex gives us two 2-simplicies σ and τ in

$$\delta^*_{2,\{2\}} \mathrm{Fun}(\Delta^1, \mathrm{Sch}_{\mathsf{fd}})^{\mathrm{cart}}_{\mathrm{F,ALL}}$$

which are morphisms

$$\Delta^2 \times (\Delta^2)^{op} \times \Delta^1 \to \mathrm{Sch}_{fd}$$

whose upper 3×3 grids are:

(a)

$$\sigma|_{\Delta^2\times\Delta^2\times\{0\}} := \begin{array}{ccccc} Y' & \longleftarrow & Y' & \xleftarrow[f']{} & X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Y & \longleftarrow & Y & \xleftarrow[f]{} & X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & Y & \xleftarrow[f]{} & X \end{array} \tag{4.23}$$

(b)

$$\tau|_{\Delta^2\times\Delta^2\times\{0\}} := \begin{array}{ccccc} Y' & \xleftarrow[f']{} & X' & \longleftarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \xleftarrow[f]{} & X' & \longleftarrow & X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g'} \\ Y & \xleftarrow[f]{} & X & \longleftarrow & X \end{array} \tag{4.24}$$

As $d_1^2(\sigma) = d_1^2\tau) = \gamma$,
We have

$$\mathrm{EO}(\mathcal{T}^{\otimes})(\gamma) \cong \mathrm{EO}(\mathcal{D}^{\otimes})^*(g) \circ \mathrm{EO}(\mathcal{D}^{\otimes})_!(f) = (\mathrm{id}, g^* \circ f_!)$$

and

$$\mathrm{EO}(\mathcal{T}^{\otimes})(\gamma) \cong \mathrm{EO}(\mathcal{D}^{\otimes})_!(f') \circ \mathrm{EO}(\mathcal{D}^{\otimes})^*(g') = (\mathrm{id}, f'_! \circ g'^*).$$

Restricting to the second coordinate, we have

$$f_! \circ g^* \cong g'^* \circ f'_!$$

which is the base change formalism.

4.4 Proof of Theorem 4.1.1.

Let us denote the collection of squares (i.e. morphisms in $\mathrm{Fun}(\Delta^1, \mathrm{N}^{\mathrm{D}}_{\bullet}(\text{Nis-locSt}))$)

$$\begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{f} & \mathcal{X}_1 \\ \downarrow & & \downarrow \\ \mathcal{Y}_0 & \xrightarrow{f'} & \mathcal{Y}_1 \end{array} \tag{4.25}$$

where f and f' are separated of finite type by F'. The idea of the proof of Theorem 4.1.1 is the following:
Suppose we construct a morphism

$$\mathrm{EO}(\mathcal{SH}^{\otimes}_{\mathrm{ext}}) : \delta^*_{2,\{2\}}\,\mathrm{Fun}(\Delta^1, \mathrm{N}^{\mathrm{D}}_{\bullet}(\text{Nis-locSt}))_{F',\mathrm{ALL}} \to \mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}}) \tag{4.26}$$

which extends $\mathrm{EO}(\mathcal{SH}^{\otimes})$, then the discussion in Section 4.3.3 gives us the lower shriek functor, projection formula and base change for our functor $\mathcal{SH}^{\otimes}_{\mathrm{ext}}$. Thus proving Theorem 4.1.1 is reduced to extending the functor $\mathrm{EO}(\mathcal{SH}^{\otimes})$ to Nis-locSt. We extend the functor in a two step process as we did for extending $\mathcal{SH}$ from schemes to algebraic stacks. Thus we formulate a proposition in setting of category of stacks admitting $\mathcal{T}$-local sections (Definition 3.2.1).

4.4.1 Extending the EO map from schemes to algebraic stacks.

As explained before, we prove the proposition which helps us to extend the enhanced operation map from schemes to algebraic stacks. The proposition is a special case of the DESCENT program stated in [LZ17, Theorem 4.1.8]). Our proof is inspired from the proof of Liu and Zheng and give a new proof of the theorem. Before stating the proposition, let us fix notations in the context of category of stacks admitting $\mathcal{T}$-local sections.

Notation 4.4.1. Let $\mathcal{E}'$ be a collection of edges in $\mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}})$ which are representable in $\mathcal{C}$, stable under pullback and compositions. We denote the collection of edges in $\mathcal{E}'$ which are in $\mathcal{C}$ by $\mathcal{E}$.
We shall denote the collection of commutative squares

$$\begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{\mathsf{f}} & \mathcal{X}_1 \\ \downarrow{\scriptstyle \nu'} & & \downarrow \\ \mathcal{Y}_0 & \xrightarrow{\mathsf{f}'} & \mathcal{Y}_1 \end{array} \tag{4.27}$$

where $\mathsf{f},\mathsf{f}' \in \mathcal{E}'$ by F'. The collection of all such pullback squares in $\mathcal{C}$ shall be denoted by F. We assume that there exists a functor

$$\mathrm{EO}(\mathcal{D}^{\otimes}) : \mathrm{EO}(\mathcal{C}) := \delta^{*}_{2,\{2\}}\,\mathrm{Fun}(\Delta^1, \mathsf{N}(\mathcal{C}))^{\mathsf{cart}}_{\mathsf{F},\mathrm{ALL}} \to \mathrm{Mod}(\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}})$$

which when restricted to the direction ALL gives an ∞-sheaf

$$\mathcal{D}'^{\otimes} : \mathrm{Fun}(\Delta^1, \mathsf{N}(\mathcal{C}))^{\mathsf{op}} \to \mathrm{Mod}(\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}})$$

with respect to the topology induced by $\mathcal{T}$ on the functor category in a canonical way (the coverings on a object in $\mathrm{Fun}(\Delta^1, \mathsf{N}(\mathsf{C}))$ are given by commutative squares in C where the vertical arrows are coverings).

Proposition 4.4.2. *The functor* $\mathrm{EO}(\mathcal{D}^{\otimes})$ *extends to a functor*

$$\mathrm{EO}(\mathcal{D}^{\otimes}_{\mathrm{ext}}) : \mathrm{EO}(\mathrm{St}_{\mathcal{C}}) := \delta^{*}_{2,\{2\}}\,\mathrm{Fun}(\Delta^1, \mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}}))_{\mathsf{F}',\mathrm{ALL}} \to \mathrm{Mod}(\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}). \tag{4.28}$$

Idea of the proof: The proof is similar to the proof of Theorem 3.4.1.

We shall denote the simplicial set $\delta^{*}_{2,\{2\}}\,\mathrm{Fun}(\Delta^1, \mathrm{Cov}(\mathrm{St}_{\mathcal{C}}))_{\mathsf{F}',\mathrm{ALL}}$ by $\mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})$. We also denote the projection map

$$\mathrm{EOCov}(\mathrm{St}_{\mathcal{C}}) \to \mathrm{EO}(\mathrm{St}_{\mathcal{C}})$$

induced by the map $\mathsf{p} : \mathrm{Cov}(\mathrm{St}_{\mathcal{C}}) \to \mathsf{N}^{\mathsf{D}}_{\bullet}(\mathrm{St}_{\mathcal{C}})$ by p_{EO}.
The following claim implies that the morphism p_{EO} is surjective on every simplex.

Claim 4.4.3. Let τ_n be a morphism $\tau_n : \Delta^n \times \Delta^n \to N^D_\bullet(\mathrm{St}_{\mathcal{C}})$ which is an n-simplex of $\delta^*_{2,\{2\}}(N^D_\bullet(\mathrm{St}_{\mathcal{C}}))^{cart}_{F',\mathrm{ALL}}$. Then there exists a map

$$\tau^1_n : \Delta^1 \times \Delta^n \times \Delta^n \to N^D_\bullet(\mathrm{St}_{\mathcal{C}})$$

such that

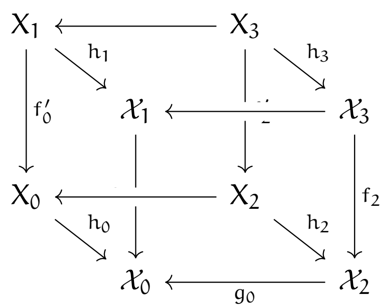
1. $\tau^1_n|_{[0]\times\Delta^n\times\Delta^n}$ is a n-simplex of $\delta^*_{2,\{2\}}N(\mathcal{C})^{cart}_{F,\mathrm{ALL}}$,
2. $\tau^1_n|_{[1]\times\Delta^n\times\Delta^n} = \tau_n$ and
3. $\tau^1_n : [k] \times [i] \times [j] :\to N^D_\bullet(\mathrm{St}_{\mathcal{C}})$ is a Čech nerve of a morphism admitting $\mathcal{T}$-local sections for all $0 \le i, j \le n$ and $0 \le k \le 1$.

The claim implies surjectivity of p_{EO} because one can apply the claim to upper and lower pullback squares of any n-simplex of $\delta^*_{2,\{2\}}\,\mathrm{Fun}(\Delta^1, N^D_\bullet(\mathrm{St}_{\mathcal{C}}))^{cart}_{F,\mathrm{ALL}}$. By taking fiber products, this produces an edge in $\mathrm{Fun}((\Delta^n)^{op} \times \Delta^n, \mathrm{Fun}(\Delta^1, N^D_\bullet(\mathrm{St}_{\mathcal{C}})$. Considering the Čech nerve of the edge gives us an n-simplex of $\mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})$.

Proof. We prove it for $n = 1$. The case of higher n follows from induction choosing a compatible choice of atlas. We want to show that for a pullback square of the form

$$\begin{array}{ccc} \mathcal{X}_1 & \xleftarrow{g_1} & \mathcal{X}_3 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_2} \\ \mathcal{X}_0 & \xleftarrow{g_0} & \mathcal{X}_2 \end{array} \tag{4.29}$$

in $\mathrm{St}_{\mathcal{C}}$ where f_0 and f_2 are in $\mathcal{E}'$, there exists a cube of the form



(4.30)

where the square formed by vertices of X_0, X_1, X_2 and X_3 is a pullback square, $f'_0, f'_2 \in \mathcal{E}$ and h_0, h_1, h_2, h_3 are atlases admitting $\mathcal{T}$-local sections.

Let $h_0 : X_0 \to \mathcal{X}_0$ be an atlas admitting $\mathcal{T}$-local sections. By Claim 3.4.2, there exists a commutative square of the form

$$\begin{array}{ccc} X_0 & \longleftarrow & X_2 \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_2} \\ \mathcal{X}_0 & \longleftarrow & \mathcal{X}_2 \end{array} \tag{4.31}$$

where h_0 and h_2 are atlases admitting $\mathcal{T}$-local sections. As f_0 is representable, the base change morphism $h_1 : X_1 := X_0 \times_{\mathcal{X}_0} \mathcal{X}_1 \to \mathcal{X}_1$ is an atlas admitting $\mathcal{T}$-local sections. Also $f'_0 : X_1 \to X_0$ lies in $\mathcal{E}$. Then defining $X_3 := X_1 \times_{X_0} X_2$ gives us the cube that we wanted.

□

We shall denote the collection of edges in $\mathrm{Fun}(\Delta^1, \mathrm{Cov}(\mathrm{St}_{\mathcal{C}}))$ (i.e. morphisms of the form $\Delta^1 \times \Delta^1 \to \mathrm{St}_{\mathcal{C}}$) which are of the form:

$$\begin{array}{ccc} (\mathcal{X}, x : X \to \mathcal{X}) & \xrightarrow{f} & (\mathcal{X}, x' : X' \to \mathcal{X}) \\ \downarrow & & \downarrow \\ (\mathcal{Y}, y : Y \to \mathcal{Y}) & \xrightarrow{f'} & (\mathcal{Y}, y' : Y' \to \mathcal{Y}) \end{array} \tag{4.32}$$

by R'. We shall also denote the simplicial set $\delta^*_{2,\{2\}} \mathrm{Fun}(\Delta^1, \mathrm{Cov}(\mathrm{St}_{\mathcal{C}}))[R^{-1}]^{cart}_{F',\mathrm{ALL}}$ by $\mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})[R'^{-1}]$. We have the following claim.

Claim 4.4.4. The map

$$p'_{EO} : \mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})[R'^{-1}] \to \mathrm{EO}(\mathrm{St}_{\mathcal{C}})$$

induced by the map $p'' : \mathrm{Fun}(\Delta^1, \mathrm{Cov}(\mathrm{St}_{\mathcal{C}}))[R^{-1}] \to \mathrm{Fun}(\Delta^1, N^D_\bullet(\mathrm{St}_{\mathcal{C}}))$ is categorical equivalence of simplicial sets.

Proof. Following the arguments in the proof of Theorem 3.4.1, we see that the map p'' is a categorical equivalence. Thus it admits a categorical inverse q''. It can be easily verified that q'' sends pullback squares to pullback squares. Thus the map q'' induces a map

$$q'_{EO} : \mathrm{EO}(\mathrm{St}_{\mathcal{C}}) \to \mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})[R'^{-1}].$$

As $q'' \circ p'' = \mathrm{id}_{\mathrm{Fun}(\Delta^1, \mathrm{St}_{\mathcal{C}})}$, we have $p'_{EO} \circ q'_{EO} = \mathrm{id}_{\mathrm{EO}(\mathrm{St}_{\mathcal{C}})}$. On the other hand, we see that $q'_{EO} \circ p'_{EO}$ is preisomorphic to $\mathrm{id}_{\mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})[R'^{-1}]}$ in the sense of [Rez, 21.4]. Thus by [Rez, Lemma 21.8], we get that p'_{EO} is a categorical equivalence. □

We also prove another claim regarding the monomorphism

$$i_{EO} : \mathrm{EOCov}(\mathrm{St}_{\mathcal{C}}) \hookrightarrow \mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})[R'^{-1}].$$

Claim 4.4.5. Let $\mathcal{E}$ be an ∞-category and let $F : \mathrm{EOCov}(\mathrm{St}_{\mathcal{C}}) \to \mathcal{E}$ be a functor which maps R' to equivalences. Then F extends to a functor $F' : \mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})[R'^{-1}] \to \mathcal{E}$.

Proof. We construct the functor F' inductively. As objects of $\mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})[R'^{-1}]$ are same as objects of $\mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})$, we define F' as F.
Assume we have defined F' upto $n-1$ simplices. Let σ_n be an n-simplex of $\mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})$. By induction, the boundary of σ_n maps to a morphism $F'(\partial(\sigma_n)) : \partial\Delta^n \to \mathcal{D}$.
We denote the image of σ_n via p'_{EO} by σ'_n. As p_{EO} is surjective on each simplex, this lifts to an element $\sigma''_n : \Delta^n \to \mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})$. Following the arguments of Proposition 3.4.3, we get a morphism

$$\tau_n : \partial\Delta^n \times \Delta^1 \coprod_{\partial\Delta^n \times \{0\}} \Delta^n \to \mathcal{D}$$

where

1. $\tau_n|_{[0]\times\Delta^n} = F(\sigma''_n)$.

2. $\tau_n|_{[1]\times\partial\Delta^n} = F'(\partial(\sigma_n))$, and

3. $\tau_n|_{\Delta^1\times[k]}$ is an equivalence for all $0 \leq k \leq n$.

As explained in Proposition 3.4.3, this morphism extends to a morphism

$$\tau'_n : \Delta^n \times \Delta^1 \to \mathcal{D}.$$

The morphism τ'_n when restricted to $\Delta^n \times [1]$ gives us the morphism

$$F'(\sigma_n) : \Delta^n \to \mathcal{D}.$$

This completes the proof of induction and hence the claim. □

Proof of Theorem 4.1.1. We define a morphism

$$\varphi : \mathrm{EOCov}(\mathrm{St}_{\mathcal{C}}) \xrightarrow{\mathrm{EO}(\mathcal{D}^{\otimes})} \mathrm{Fun}(N(\Delta), \mathrm{Mod}(\mathrm{Pr}^{L}_{\mathrm{stb}})) \xrightarrow{\mathrm{res}_{[-1]}\circ i} \mathrm{Mod}(\mathrm{Pr}^{L}_{\mathrm{stb}}) \tag{4.33}$$

as follows:

1. Note that we have a canonical morphism

$$\mathrm{EOCov}(\mathrm{St}_{\mathcal{C}}) \to \mathrm{Fun}(N(\Delta), \mathrm{EO}(\mathcal{C})).$$

   The morphism $\mathrm{EO}(\mathcal{D}^{\otimes})$ is the functor $\mathrm{EO}(\mathcal{D}^{\otimes})$ applied to $\mathrm{Fun}(N(\Delta), \mathrm{EO}(\mathcal{C}))$.

2. The maps i and $\mathrm{res}_{[-1]\times\Delta^n}$ are the same maps that we defined in the construction of ϕ in Theorem 3.4.1. The functorial association of limit is possible as $\mathrm{Mod}(\mathrm{Pr}^{L}_{\mathrm{stb}})$ admits small limits (Theorem B.2.8).

Similar to the arguments in Theorem 3.4.1, we see that the morphism φ sends R to equivalences. By Claim 4.4.5, this induces a map

$$\varphi' : \mathrm{EOCov}(\mathrm{St}_{\mathcal{C}})[R^{-1}] \to \mathrm{Mod}(\mathrm{Pr}^{L}_{\mathrm{stb}}).$$

By Claim 4.4.4, this induces a morphism

$$\mathrm{EO}(\mathcal{D}^{\otimes}_{\mathrm{ext}}) : \mathrm{EO}(\mathrm{St}_{\mathcal{C}}) \to \mathrm{Mod}(\mathrm{Pr}^{L}_{\mathrm{stb}}).$$

It is automatically clear that this an extension of the morphism $\mathrm{EO}(\mathcal{D}^{\otimes})$. This completes the proof. □

Remark 4.4.6. The functor $\mathrm{EO}(\mathcal{D}^{\otimes}_{\mathrm{ext}})$ when restricted to direction ALL is indeed the functor $\mathcal{D}'^{\otimes}_{\mathrm{ext}}$ obtained by applying Theorem 3.4.1 to functor $D'^{\otimes}$).

4.4.2 Conclusion of proof of Theorem 4.1.1.

To conclude the proof, we apply Proposition 4.4.2 to the functor $\mathrm{EO}(\mathcal{SH}^{\otimes})$ and F' to be collection of morphisms which are representable, separated and finite type. We first extend the functor to the category of algebraic spaces. Then, we apply it again to extend it to the $(2,1)$-category Nis-locSt.

Thus this gives us the enhanced operation map

$$\mathrm{EO}(\mathcal{SH}^{\otimes}_{\mathrm{ext}}) : \delta^*_{2,\{2\}} \mathrm{Fun}(\Delta^1, \mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt}))^{\mathsf{cart}}_{\mathsf{F}',\mathrm{ALL}} \to \mathrm{Mod}(\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}). \tag{4.34}$$

The composition

$$\mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt})^{\mathsf{op}} \xrightarrow{\mathrm{dir}_{\mathrm{ALL}} \circ (\mathcal{X} \to (\mathcal{X} \to \mathrm{Spec}\, \mathbf{Z}))} \delta^*_{2,\{2\}} \mathrm{Fun}(\Delta^1, \mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt}))^{\mathsf{cart}}_{\mathsf{F}',\mathrm{ALL}} \xrightarrow{\mathrm{EO}(\mathcal{SH}^{\otimes}_{\mathrm{ext}})} \mathrm{Mod}(\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}) \tag{4.35}$$

sends $\mathcal{X}$ to $(\mathcal{SH}^{\otimes}_{\mathrm{ext}}(\mathbf{Z}), \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}))$ and it sends morphisms $\mathsf{f} : \mathcal{X} \to \mathcal{Y}$ to the pair of morphisms $(\mathrm{id}, \mathsf{f}^*) : (\mathcal{SH}^{\otimes}_{\mathrm{ext}}(\mathbf{Z}), \mathcal{SH}_{\mathrm{ext}}(\mathcal{Y})) \to (\mathcal{SH}^{\otimes}_{\mathrm{ext}}(\mathbf{Z}), \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}))$.

As explained in Section 4.3.3, the functor $\mathrm{EO}(\mathcal{SH}^{\otimes}_{\mathrm{ext}})$ induced by restricting along direction F' gives us the functor

$$\mathcal{SH}_{\mathrm{ext}\,!} : \mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt}') \to \mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}\,. \tag{4.36}$$

which extends $\mathcal{SH}_!$.

The projection and base change formulas also follow from $\mathrm{EO}(\mathcal{SH}^{\otimes}_{\mathrm{ext}})$ as explained in beginning of Section 4.4 and Section 4.3.3. This completes the proof of Theorem 4.1.1.

CHAPTER 5

SIX OPERATIONS FOR SH($\mathcal{X}$)

In the previous two chapters, we have extended the motivic homotopy functor from schemes to algebraic stacks and constructed the six functors. In this chapter, we prove other relations of six operations: homotopy invariance, localization and purity.

In the first section, we state results of smooth and proper base change theorems in our context. In the second section, we prove the theorems of localization and homotopy invariance. In the third section, we construct the natural transformation α_f. In the fourth section, we construction the purity transformation ρ_f. In the last section, we summarize all the results and state in a single theorem.

5.1 Smooth and proper base change.

In this section, we prove smooth and proper base change theorems. Theorem 4.1.1 constructs the lower shriek functors for representable morphisms and separated of finite type, in particular for smooth and proper morphisms. On the level of schemes, for a smooth morphism $f : X \to Y$, the pullback morphism $F^* : \mathcal{SH}(Y) \to \mathcal{SH}(X)$ admits a left adjoint $f_\#$. It is natural to expect such a result in the context of $\mathcal{SH}^{\otimes}_{\text{ext}}$ of Nis-locSt.

Lemma 5.1.1. *Let $\mathcal{X} \in$ Nis-locSt and $x : X \to \mathcal{X}$ be a smooth atlas which admits Nisnevich-local sections. Then the pullback map $x^* : \mathcal{SH}_{\text{ext}}(\mathcal{X}) \to \mathcal{SH}_{\text{ext}}(X)$ is conservative.*

Proof. This is a consequence of [Lur17, Proposition 4.7.5.1] applied to the $\mathcal{J} = N(\Delta_+)$ and $q : N(\Delta) \to \widehat{\text{Cat}_\infty}$ which is the functor $\mathcal{SH}^{\otimes}_{\text{ext}}$ applied to simplicial object $X^{\bullet}_{x}$. We get that the functor $G := x^* : \mathcal{SH}_{\text{ext}}(\mathcal{X}) \to \mathcal{SH}_{\text{ext}}(X)$ is conservative. □

Proposition 5.1.2. *(Smooth base change) Let $f : \mathcal{Y} \to \mathcal{X}$ be a representable smooth morphism in Nis-locSt. Then f^* admits a left adjoint $f_\#$. Morever for a cartesian square in Nis-locSt* *of the form*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ \downarrow{g'} & & \downarrow{g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \tag{5.1}$$

where f *(and thus* f'*) is smooth, we have an equivalence*

$$\mathrm{Ex}(\Delta^*_\#) : f^* g_\# \cong g'_\# f'^*.$$

Proof. The proof of the proposition uses the theory of left and right adjointable squares and A.13.6. Let $x : X \to \mathcal{X}$ be an atlas admitting Nisnevich-local sections. Then $y : X \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is an atlas admitting $\mathcal{T}$-local sections. Taking Čech nerves of x and y, produces a morphism

$$H : N(\Delta)^{op} \times \Delta^1 \to N^D_\bullet(\mathrm{St}_\mathcal{C})$$

where $H|_{[0]\times\Delta^1} = f'$ (where f' is base change of f along x), $H|_{N(\Delta_+)^{op}\times[0]} = X^\bullet_x$ and $H|_{N(\Delta_+)^{op}\times[1]} = Y^\bullet_y$.
Composing with the functor $\mathcal{SH}_{\mathrm{ext}} : N^D_\bullet(\mathrm{St}_\mathcal{C})^{op} \to \mathrm{Pr}^L_{\mathrm{stb}} \hookrightarrow \widehat{\mathrm{Cat}_\infty}$, we get a functor

$$H : N(\Delta) \to \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}_\infty})$$

As $f_\#$ is left adjoint of f^* on the level of schemes and we have smooth base change (see [Rob14, Example 9.4.8] which we recalled in Proposition C.4.1), this implies that the functor H can be realized as functor:

$$H : N(\Delta) \to \mathrm{Fun}^{LAd}(\Delta^1, \widehat{\mathrm{Cat}_\infty})$$

where $\mathrm{Fun}^{LAd}(\Delta^1, \widehat{\mathrm{Cat}_\infty})$ is the ∞-category of left adjointable functors ([Lur17, Definition 4.7.4.13]). As $\mathrm{Fun}^{LAd}(\Delta^1, \widehat{\mathrm{Cat}_\infty})$ admits small limits ([Lur17, Corollary 4.7.4.18]), the map H admits a limit

$$\overline{H} : N(\Delta_+)^{op} \to \mathrm{Fun}^{LAd}(\Delta^1, \widehat{\mathrm{Cat}_\infty}).$$

Evaluating H' at $[-1]$, we get the morphism

$$f^* : \mathcal{SH}_{\mathrm{ext}}(\mathcal{Y}) \to \mathcal{SH}_{\mathrm{ext}}(\mathcal{X})$$

which is an element of $\mathrm{Fun}^{LAd}(\Delta^1, \widehat{\mathrm{Cat}_\infty})$. By definition of ∞-category of left adjointable functors, we get that f^* admits a left adjoint

$$f_\# : \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}) \to \mathcal{SH}_{\mathrm{ext}}(\mathcal{Y}).$$

It remains to prove the smooth base change. Let us denote the cartesian square in the proposition as a morphism $\sigma : \Delta^1 \times \Delta^1 \to N^D_\bullet(\mathrm{St}_\mathcal{C})$ Let $y : Y \to \mathcal{Y}$ be an atlas admitting $\mathcal{T}$-local sections. By Claim 3.4.2 and the fact that pullback of representable smooth morphism is smooth, we get a morphism

$$G' : N(\Delta_+)^{op} \times \Delta^1 \times \Delta^1 \to \mathcal{C}$$

such that

1. $G'|_{[-1]\times\Delta^1\times\Delta^1} = \sigma$.

2. $G'_{N(\Delta_+)^{op}\times[k]\times[j]}$ is Čech nerve of an atlas of $\sigma([k],[j])$ admitting Nisnevich-local sections for all $0 \leq j, k \leq 1$.

Composing with $\mathcal{SH}(-)$, we get a functor

$$G : N(\Delta) \times \Delta^1 \times \Delta^1 \to \widehat{\mathrm{Cat}_\infty}.$$

For every $\tau : \Delta^1 \to N(\Delta) \times \Delta^1$, the induced square

$$G \circ (\tau \times \Delta^1) : \Delta^1 \times \Delta^1 \to \widehat{\mathrm{Cat}_\infty}$$

is left adjointable by smooth base change theorem on the level of schemes ([Rob14, Example 9.4.8]). Applying [LZ17, Lemma 4.3.7] to the functor G, we get that the square

$$\begin{array}{ccc} \mathcal{SH}_{\mathrm{ext}}(\mathcal{Y}) & \xrightarrow{f^*} & \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}) \\ \downarrow g^* & & \downarrow g'^* \\ \mathcal{SH}_{\mathrm{ext}}(\mathcal{Y}') & \xrightarrow{f'^*} & \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}') \end{array} \tag{5.2}$$

is the left adjointable, i.e.. we have

$$\mathrm{Ex}(\Delta^*_\#) : g'_\# f'^* \cong f^* g_\#.$$

□

Remark 5.1.3. The lower shriek functor on the level of schemes agrees with $(-)_\#$ for open immersions and $(-)_*$ along proper morphisms ([Rob14, Theorem 9.4.8]). As the lower shriek functor on the level of algebraic stacks is constructed by taking limit along Čech covers of atlases, we get that for an open immersion $j : \mathcal{X}' \to \mathcal{X}$, we have an equivalence $j_! \cong j_\#$ and for representable proper morphisms $p : \mathcal{X}'' \to \mathcal{X}$, we have an equivalence $p_! \cong p_*$.

A similar proposition holds in the case of representable proper morphisms.

Proposition 5.1.4 (Proper base change)**.** *Given a cartesian square in* Nis-locSt *of the form*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ \downarrow g' & & \downarrow g \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \tag{5.3}$$

where g (and thus g') is representable and proper, we have an equivalence

$$\mathrm{Ex}(\Delta^*_*) : f^* g_* \cong g'_* f'^*.$$

Proof. As $f_* \cong f_!$ when f is representable and proper, the proper base change is indeed the base change with respect to lower shriek functor. This holds due to Theorem 4.1.1. □

5.2 Localization and homotopy invariance.

In this section we prove the localization theorem and homotopy property.

Proposition 5.2.1. *(Localization) If* $i : \mathcal{Z} \to \mathcal{X}$ *is a closed immersion with complementary open immersion* $j : U := X - Z \hookrightarrow \mathcal{X}$*, we have the cofiber sequences:*

1.
$$j_! j^! \to \mathrm{id} \to i_* i^* \tag{5.4}$$

2.
$$i_! i^! \to \mathrm{id} \to j_* j^* \tag{5.5}$$

Proof. We prove the existence of the first cofiber sequence. The second cofiber sequence is dual to the first one. Let $x : X \to \mathcal{X}$ be an atlas admitting Nisnevich-local sections.
Then the restriction of x to $\mathcal{Z}$ and $\mathcal{U}$ defines atlases $z : Z \to \mathcal{Z}$ and $u : U \to \mathcal{U}$ and these induce morphism of the Čech nerves $Z^+_{\bullet,z} \to X^+_{\bullet,x}$ and $U^+_{\bullet,u} \to X^+_{\bullet,x}$. Thus we have morphisms $Z^n_{\mathcal{Z}} \hookrightarrow X^n_{\mathcal{X}}$ and $U^n_{\mathcal{U}} \hookrightarrow X^n_{\mathcal{X}}$ which are closed and open immersions respectively for every n.
Let $E \in \mathcal{SH}_{\mathrm{ext}}(\mathcal{X})$, then $E = (E_n)_{n\in\Delta}$ where $E_n \in \mathcal{SH}(X^n_{\mathcal{X}})$.

For any n, we have a square which is a fiber sequence

$$\begin{array}{ccc} j_{n\#}j_n^*(E_n) & \longrightarrow & E_n \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_{n*}i_n^*(E_n) \end{array} \tag{5.6}$$

in $\mathcal{SH}(X^n_{\mathcal{X}})$, because localization holds for schemes ([Rob14, Theorem 9.4.25]).

This can be visualized as a limit map

$$\overline{H_n} : \Lambda_0^{2^{\triangleleft}} \to \mathrm{Cat}_\infty$$

where

$$H_n : \Lambda^2_0 \to \widehat{\mathrm{Cat}_\infty}$$

Also, we have a morphism $H_{-1} : \Lambda^2_0 \to \widehat{\mathrm{Cat}_\infty}$ given by the diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow \\ 0 & \longrightarrow & i_*i^*E \end{array} \tag{5.7}$$

The collection of maps H_n for every $n \geq -1$ induces a morphism

$$H : \Lambda^2_0 \to \mathrm{Fun}(N(\Delta_{s+}), \widehat{\mathrm{Cat}_\infty})$$

where for every $[n] \in \Delta_{s+}$, $H|_{[n]} := H_n$. As $\mathrm{Fun}(N(\Delta_{s+}), \widehat{\mathrm{Cat}_\infty})$ admits all limits, there exists an extension of H to

$$\overline{H} : \Delta^1 \times \Delta^1 \cong \Lambda_0^{2^{\triangleleft}} \to \mathrm{Fun}(N(\Delta_{s+}), \widehat{\mathrm{Cat}_\infty}).$$

The morphism $\overline{H}$ is evaluated at $[-1] \in \Delta_{s+}$ gives us a square

$$\overline{H}|_{[-1]} : \Delta^1 \times \Delta^1 \to \widehat{\mathrm{Cat}_\infty}.$$

As smooth pullbacks commute with $(-)_\#, (-)_*$, we have $j_\# j^*(E) \cong (j_{n\#} j_n^*(E_n))_{n\in\Delta}$ and $i_* i^*(E) \cong (i_{n*} I_n^*(E_n))_{n\in\Delta}$. Thus the morphism $H'|_{[-1]}$ is the pullback square

$$\begin{array}{ccc} j_\# j^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_* i^*(E). \end{array} \tag{5.8}$$

This proves the localization theorem.
The same argument works for the dual sequence 2. □

Proposition 5.2.2. *(Homotopy invariance) For any stack $\mathcal{X} \in$ Nis-locSt, the projection $\pi : \mathbb{A}^1_{\mathcal{X}} \to \mathcal{X}$ induces a fully faithful functor* $\mathrm{pi}^*(-) : \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}) \to \mathcal{SH}_{\mathrm{ext}}(\mathbb{A}^1_{\mathcal{X}})$

Proof. To show that $\pi^*_{\mathcal{X}}$ is fully faithful, we need to show that the unit transformation $u : \pi_\# \pi^* \to \mathrm{id}$ is an equivalence. As $\pi_\#$ satisfies projection formula, we are reduced to showing $u(1_{\mathcal{X}}) : \pi_\# \pi^*(1_{\mathcal{X}}) \to 1_{\mathcal{X}}$ is an equivalence.

Fixing the usual atlas $x : X \to \mathcal{X}$, let $\pi_0 : \mathbb{A}^1_X \to X$ be the projection map. As $\mathcal{SH}$ satisfies homotopy invariance on the level of schemes, we have an equivalence $u_0(1_X) : \pi_{0\#}\pi_0^*(1_X) \to 1_X$. As $(-)_\#$ commutes with pullbacks, we get that pullback of $u(1_{\mathcal{X}})$ along x^* is $u_0(1_X)$. As x^* is conservative (Lemma 5.1.1) and $u_0(1_X)$ is an equivalence, we get that $u(1_{\mathcal{X}})$ is an equivalence. □

5.3 The natural transformation α_f.

In this section, we construct the natural transformations α_f and which is extensions of the natural transformation of the same notation on the level of schemes ([CD19, Proposition 2.2.10]). We construct the natural transformation for a specific class of morphisms in Nis-locSt.

Definition 5.3.1. A representable morphism $f : \mathcal{X} \to \mathcal{Y}$ in Nis-locSt is *compactifiable* if admits a factorization of the form $\mathcal{X} \xrightarrow{j} \overline{\mathcal{X}} \xrightarrow{p} \mathcal{Y}$ where j is an open immersion and p is a proper representable morphism of algebraic stacks.

Example 5.3.2. Open immersions and representable proper morphisms are compactifiable.

Proposition 5.3.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a compactifiable morphism of algebraic stacks in* Nis-locSt*. Then there exists a natural transformation:*

$$\alpha_f : f_! \to f_* \tag{5.9}$$

which is an equivalence if f is proper.

Proof. The construction of α_f is similar to the construction on the level of schemes (Appendix C.4.3). Consider a factorization (j, p) of the f.
At first, we have a natural transformation

$$j_\# \to j_*$$

for any open immersion j. This follows because at first $j^* j_\# \cong \mathrm{id}$ by smooth base change $\mathrm{Ex}(\Delta^*_\#)$ applied to the cartesian square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\mathrm{id}} & \mathcal{X} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle j} \\ \mathcal{X} & \xrightarrow{j} & \overline{\mathcal{X}} \end{array} \tag{5.10}$$

Then by adjoint property (Proposition 5.1.2), we have a morphism $j_\# \to j_*$.

Thus the natural transformation α_f is defined as

$$\alpha_f : f_! = p_* j_\# \to p_* j_* \cong f_*.$$

When f is proper, the definition of compactifiable implies that α_f is an equivalence.

□

5.4 Homotopy purity.

In this section we prove the homotopy purity theorem of $\mathcal{SH}^{\otimes}_{\mathrm{ext}}(-)$. At first, we construct the natural transformation ρ_f which is analog to the purity transformation on the level of schemes. We then prove the homotopy purity theorem using the deformation to the normal cone. Also as a corollary, we get an explicit description of the self equivalence Tw_f.

Proposition 5.4.1. *(Purity) Let $f : \mathcal{X} \to \mathcal{Y}$ to be smooth morphism separated of finite type, there exists a self equivalence $\mathrm{Tw}_f : \mathcal{SH}_{\mathrm{ext}}(\mathcal{X}) \to \mathcal{SH}_{\mathrm{ext}}(\mathcal{X})$ and an equivalence*

$$\mathrm{Tw}_f \circ f^! \cong f^*.$$

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth morphism separated of finite type. Let $\delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be the diagonal morphism and $p : \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ be the projection map. Then we denote the *Thom transformation*

$$\Sigma_f := p_\# \circ \delta_*.$$

By the base change theorem with respect to $f_!$ (Theorem 4.1.1) and the fact that α_f is an equivalence for proper morphisms (Proposition 5.3.3), we can construct the transformation as one does on the level of schemes (Appendix C.4.3)

$$\rho_f : f_\# \to f_! \circ \Sigma_f.$$

Let us check that Σ_f and ρ_f are equivalences. When base changed to the level of schemes by choosing an atlas, these natural transformations are equivalences ([Rob14, Theorem 9.4.37]). As the pullback functor along an atlas is conservative (Lemma 5.1.1), this implies that these natural transformations are equivalences.

□

The above proposition gives us the relation between f^* and $f^!$ via the self equivalence $\mathrm{Tw}_f := (\Sigma_f)^{-1}$. We want to have a precise description of the self equivalence Σ_f as one gets on the level of schemes via deformation to normal cone. We prove a similar result in the context of $\mathcal{SH}^{\otimes}_{\text{ext}}(-)$.
At first we define the notion of smooth closed pairs in this context as one does on the level of schemes.

Definition 5.4.2. Let $\mathcal{Y}$ be an algebraic stack in $\mathsf{N}^{\mathsf{D}}_{\bullet}$(Nis-locSt). A *smooth closed pair* over $\mathcal{Y}$ is a pair $(\mathcal{X}, \mathcal{Z})$ where:

1. $\mathcal{X}, \mathcal{Z}$ are stacks over $\mathcal{Y}$ in $\mathsf{N}^{\mathsf{D}}_{\bullet}$(Nis-locSt) such that the projection maps to $\mathcal{Y}$ are smooth.
2. $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed substack of $\mathcal{X}$.

A morphism of smooth closed pairs $(\mathcal{X}, \mathcal{Z}) \to (\mathcal{X}', \mathcal{Z}')$ is a representable morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{X}'$ such that $f^{-1}(\mathcal{Z}') = \mathcal{Z}$ as a set.

Notation 5.4.3. 1. For a smooth closed pair $(\mathcal{X}, \mathcal{Z})$ over $\mathcal{Y}$, we denote

$$\frac{\mathcal{X}}{\mathcal{X} - \mathcal{Z}} := p_{\#}(i_{\mathcal{Z}*}(1_{\mathcal{X}})) \in \mathcal{SH}^{\otimes}_{\text{ext}}(\mathcal{Y}) \tag{5.11}$$

where $p : \mathcal{X} \to \mathcal{Y}$.

2. Let $p : \mathcal{V} \to \mathcal{Y}$ be a vector bundle over the algebraic stack $\mathcal{Y}$ where $\mathcal{V}, \mathcal{Y} \in \mathsf{N}^{\mathsf{D}}_{\bullet}$(Nis-locSt). Let $s : \mathcal{Y} \to \mathcal{V}$ be the zero section. Then the pair is $(\mathcal{V}, \mathcal{Y})$ a smooth closed pair where $\mathcal{Y}$ is realized as a closed substack of $\mathcal{V}$ via the zero section.

Remark 5.4.4. A morphism of smooth closed pairs $(\mathcal{X}, \mathcal{Z}) \to (\mathcal{X}', \mathcal{Z}')$ induces a map

$$\frac{\mathcal{X}}{\mathcal{X} - \mathcal{Z}} \to \frac{\mathcal{X}'}{\mathcal{X}' - \mathcal{Z}'} \quad \text{(see [CD19, 2.4.32]).}$$

We now state and prove the homotopy purity theorem. Recall that for a closed substack $\mathcal{Z} \hookrightarrow \mathcal{X}$, we denote the normal cone by $\mathsf{N}_{\mathcal{Z}}(\mathcal{X})$ and the deformation to the normal cone by $\mathsf{D}_{\mathcal{Z}}(\mathcal{X})$ (see Notation 3.3.9).

Proposition 5.4.5. *Let $(\mathcal{X}, \mathcal{Z})$ be a smooth closed pair over $\mathcal{Y}$. Then the canonical morphisms of smooth closed pairs*

$$(\mathcal{X}, \mathcal{Z}) \overset{p_0}{\hookleftarrow} (\mathsf{D}_{\mathcal{Z}}(\mathcal{X}), \mathbb{A}^1_{\mathcal{Z}}) \xrightarrow{p_1} (\mathsf{N}_{\mathcal{Z}}\mathcal{X}, \mathcal{Z}) \tag{5.12}$$

induces an equivalence

$$\frac{\mathcal{X}}{\mathcal{X} - \mathcal{Z}} \cong \frac{\mathsf{D}_{\mathcal{Z}}\mathcal{X}}{\mathsf{D}_{\mathcal{Z}}\mathcal{X} - \mathbb{A}^1_{\mathcal{Z}}} \cong \frac{\mathsf{N}_{\mathcal{Z}}\mathcal{X}}{\mathsf{N}_{\mathcal{Z}}\mathcal{X} - \mathcal{Z}} \tag{5.13}$$

Proof. By Corollary 3.3.10, we know that the algebraic stacks $\mathsf{D}_{\mathcal{Z}}(\mathcal{X})$ and $\mathsf{N}_{\mathcal{Z}}(\mathcal{X})$ are in Nis-locSt when $\mathcal{X} \in$ Nis-locSt.
The morphisms of smooth of closed pairs in $\mathsf{N}^{\mathsf{D}}_{\bullet}$(Nis-locSt)

$$(\mathcal{X}, \mathcal{Z}) \overset{p_0}{\hookleftarrow} (\mathsf{D}_{\mathcal{Z}}(\mathcal{X}), \mathbb{A}^1_{\mathcal{Z}}) \xrightarrow{p_1} (\mathsf{N}_{\mathcal{Z}}\mathcal{X}, \mathcal{Z}) \tag{5.14}$$

on base changed to an atlas $y : Y \to \mathcal{Y}$ yields us morphisms

$$(X, Z) \xleftarrow{p_{0,0}} (D_Z(X), \mathbb{A}^1_Z) \xrightarrow{p_{1,0}} (N_Z(X), Z_{\mathcal{Z}}) \tag{5.15}$$

By Proposition C.4.12, we have an equivalence:

$$\frac{X}{X - Z} \underset{\cong}{\xleftarrow{p_{0,0*}}} \frac{D_Z(X)_{\mathcal{X}}}{D_Z(X) - \mathbb{A}^1_Z} \underset{\cong}{\xrightarrow{p_{1,0*}}} \frac{N_Z(X)}{N_Z(X) - Z} \tag{5.16}$$

The construction of morphisms p_{0*} commutes with pullbacks. Thus we have $y*^*(p_{0*}) \cong p_{0,0*}$ and $y^*(p_{1*}) \cong p_{0,1*}$. By [Rob14, Theorem 9.4.34], we see that $p_{0,0*}$ and $p_{1,0*}$ are equivalences. As y^* is conservative (Lemma 5.1.1), we get that p_{0*} and p_{1*} are equivalences. □

The above equivalence gives us an explicit description of the Thom transformation Σ_f in terms of the Thom space of the normal bundle of f.

Notation 5.4.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth representable morphism of algebraic stacks. Then the Thom space of the normal bundle is defined as

$$\mathrm{Th}(N_f) := \frac{N_{\mathcal{X}}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})}{N_{\mathcal{X}}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) - \mathcal{Y}}.$$

This definition is analog to the one defined on the level of schemes ([Rob14, Definition 9.4.27]).

Corollary 5.4.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be separated of finite type, smooth representable morphism of algebraic stacks. Consider the commutative diagram of algebraic stacks*

$$\begin{array}{ccccc} \mathcal{X} & & & & \\ & \searrow^{\delta} & & & \\ & & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} & \xrightarrow{p} & \mathcal{X} \\ & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\ & & \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \tag{5.17}$$

Then we have:

$$\Sigma_f(-) := p_{\#}\delta_*(-) \cong - \otimes \mathrm{Th}(N_f). \tag{5.18}$$

Proof. The proof is exactly is same as one does for $\mathcal{SH}$ ([Rob14, Eq. 9.4.88]), we have

$$\begin{aligned} \Sigma_f(-) &= p_{\#}\delta_*(-) \\ &= p_{\#}\delta_*((1_{\mathcal{Y}} \otimes \delta^* p^*(-)) \\ &\cong p_{\#}(\delta_*(1_{\mathcal{Y}}) \otimes p^*(-)) \quad \text{(projection formula)} \\ &\cong p_{\#}\delta_*(1_{\mathcal{Y}}) \otimes (-) \text{(projection formula)} \\ &\cong \mathrm{Th}(N_f) \otimes (-) \text{(Proposition 5.4.5)} \end{aligned}$$

□

Remark 5.4.8. It is possible to define the Tate object $1_{\mathcal{X}}(1) \in \mathcal{SH}^{\otimes}_{\text{ext}}(\mathcal{X})$ as $\Omega^2(\mathsf{K})$ where K is defined as cofiber of the map $1_{\mathcal{Y}} \to p_{\#}p^*1_{\mathcal{X}}$ and Ω is the inverse of the suspension functor on the stable ∞-category $\mathcal{SH}^{\otimes}_{\text{ext}}(\mathcal{X})$. Then for a smooth representable morphism $f : \mathcal{Y} \to \mathcal{X}$, we have

$$\text{Th}(N_f) \cong 1_{\mathcal{Y}}(d)[2d]$$

where d is the relative dimension of the morphism f (and $1_X(d)$ is the dth iterated tensor product of $1_X(1)$). This follows from the fact that such a description holds on the level of schemes. Thus combining the description of the Thom space of normal bundle of f with the purity isomorphism, we get that:

$$f_{\#}(-) \cong f_!(1_{\mathcal{X}}(d)[2d] \otimes (-)).$$

5.5 Summarizing the results.

In this section, we summarize the results that we have proved in the previous sections in a single theorem.

Theorem 5.5.1. *The stable homotopy functor* $\mathcal{SH}^{\otimes} : \mathsf{N}(\text{Sch}_{\mathsf{fd}})^{\mathsf{op}} \to \text{CAlg}(\text{Pr}^{\mathsf{L}}_{\mathsf{stb}})$ *extends to a functor*

$$\mathcal{SH}^{\otimes}_{\text{ext}} : \mathsf{N}^{\mathsf{D}}_{\bullet}(\text{Nis-locSt})^{\mathsf{op}} \to \text{CAlg}(\text{Pr}^{\mathsf{L}}_{\mathsf{stb}}) \tag{5.19}$$

such that for a morphism $f : Y \to X$ *we have the following functors:*

1. $f^* : \mathcal{SH}^{\otimes}_{\text{ext}}(X) \to \mathcal{SH}^{\otimes}_{\text{ext}}(Y)$.
2. $f_* : \mathcal{SH}_{\text{ext}}(Y) \to \mathcal{SH}_{\text{ext}}(X)$.
3. $f_! : \mathcal{SH}_{\text{ext}}(X) \to \mathcal{SH}_{\text{ext}}(Y)$ *when* f *is representable, separated and of finite type.*
4. $f^! : \mathcal{SH}_{\text{ext}}(Y) \to \mathcal{SH}_{\text{ext}}(X)$ *when* f *is representable, separated and of finite type.*
5. $- \otimes - : \mathcal{SH}_{\text{ext}}(X) \otimes \mathcal{SH}_{\text{ext}}(X) \to \mathcal{SH}_{\text{ext}}(X)$.
6. $\text{Hom}_{\mathcal{SH}_{\text{ext}}(X)}(-,-) : \mathcal{SH}_{\text{ext}}(X) \times \mathcal{SH}_{\text{ext}}(X) \to \mathcal{SH}^{\otimes}_{\text{ext}}(X)$.

The functor $\mathcal{SH}^{\otimes}_{\text{ext}}$ *along with the functors* $(f^*, f_*, f_!, f^!, -\otimes-, \text{Hom}(-,-))$ *satisfy the following properties:*

1. *(Monoidality)* f^* *is monoidal, i.e. there exists an equivalence*
$$f^*(E \otimes E') \cong f^*(E) \otimes f^*(E') \tag{5.20}$$
for $E, E' \in \mathcal{SH}^{\otimes}_{\text{ext}}(X)$
2. *(Projection Formula) For* $E, E' \in \mathcal{SH}_{\text{ext}}(X)$ *and* $B \in \mathcal{SH}_{\text{ext}}(Y)$, *we have the following equivalences:*

   (a)
$$f_!(B \otimes f^*(E)) \cong (f_!B \otimes E) \tag{5.21}$$

(b)

$$f^! \mathrm{Hom}_{\mathcal{SH}_{\mathrm{ext}}(X)}(E, E') \cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{ext}}(Y)}(f^* E, f^! E') \tag{5.22}$$

3. *(Base Change) If*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ \downarrow{g'} & & \downarrow{g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \tag{5.23}$$

is a cartesian square of base schemes with g *being representable, separated and of finite type, we have the following equivalences:*

(a)

$$f'^* g'_! \cong g_! f^* \tag{5.24}$$

(b)

$$f'_* g'^! \cong g^! f_* \tag{5.25}$$

4. *(Proper pushforward) If* $f : \mathcal{X} \to \mathcal{Y}$ *is a compactifiable morphism, then there exists a natural transformation:*

$$\alpha_f : f_! \to f_* \tag{5.26}$$

which is an equivalence if f *is proper.*

5. *(Purity) For* f *to be representable, smooth, and separated of finite type, there exists a self equivalence* Tw_f *and an equivalence*

$$\mathrm{Tw}_f \circ f^! \cong f^*$$

6. *(Localization) For* $i : \mathcal{U} \to X$ *to be an open immersion and* $j : \mathcal{Z} := \mathcal{X} - \mathcal{U} \to \mathcal{X}$ *to be the closed immersion from the complement of* $\mathcal{U}$*, we have the cofiber sequences:*

(a)

$$j_! j^! \to \mathrm{id} \to i_* i^* \tag{5.27}$$

(b)

$$i_! i^! \to \mathrm{id} \to j_* j^* \tag{5.28}$$

7. *(Homotopy Invariance) Let* $\pi : \mathbb{A}^1_{\mathcal{X}} \to \mathcal{X}$ *be the projection map. Then* π^* *is fully faithful.*

8. $\mathcal{SH}^{\otimes}_{\mathrm{ext}}$ *satisfies descent with respect to Nisnevich-local sections in* $N^D_\bullet(\text{Nis-locSt})$.

Proof. The theorem follows from Theorem 3.4.1, Theorem 4.1.1, Proposition 5.2.1, Proposition 5.2.2, Proposition 5.3.3 and Proposition 5.4.1. □

APPENDIX A

SIMPLICIAL SETS AND ∞-CATEGORIES

We recall the notions of higher category theory in this chapter. The main references of this chapter are [Lur09], [Lur17], [Lur18b],[Lan21] and [Lur18a]. Other than recalling the theory of Kan extensions and abstract descent theory, we recall the notion of presentable stable ∞-categories.
The notions that will briefly recall in this chapter are as follows:

1. objects, morphisms and limits in ∞-categories;
2. relation between simplicial and 2-categories with ∞-categories;
3. model structure of simplicial sets;
4. Kan extensions ;
5. adjoint functors and adjointable squares ;
6. presentable ∞-categories ;
7. Čech nerves and ∞-sheaves ;
8. stable ∞-categories
9. and the ∞-category $\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}$.

A.1 Simplicial sets and simplicial objects.

Definition A.1.1. [Lur18a, Definition 1.1.1.2] Let Δ be the category defined as follows:

1. Objects: $[n] = \{0, 1, 2, \cdots, n\}$.
2. Morphisms $f : [m] \to [n]$ are maps such that $f(i) \leq f(j)$ if $i < j$.

Definition A.1.2. [Lur18a, Notation 1.1.1.8, 1.1.1.9] The category Δ has two special collection of maps for every positive integer n.

1. We have $n+1$ maps $d_i^{\prime n} : [n-1] \to [n]$ which are defined as

$$d_i^{\prime n}(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

These are called *face maps*.

2. We have n maps $s_i^{\prime n-1} : [n] \to [n-1]$ which are defined as

$$s_i^{n-1}(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

These are called *degeneracy maps*.

Definition A.1.3. [Lur18a, Definition 1.1.1.4] A *simplicial object* in a category $\mathcal{C}$ is a functor $F : \Delta^{op} \to \mathcal{C}$.
When $\mathcal{C} = \text{Sets}$, it is called *simplicial set*.

Notation A.1.4. For every n, an object of $F_n := F(n)$ is called an n-simplex.
A morphism between simplicial sets F and G is a natural transformation of functors. Denote sSets to be the category of simplicial sets.

Notation A.1.5. [Lur18a, Construction 1.1.2.1] We shall denote Δ^n as the functor:

$$\Delta^n := \text{Hom}_{\Delta}(-, [n]) : \Delta^{op} \to \text{Sets}$$

It turns out by Yoneda lemma that for any simplicial set F, we have

$$F_n = \text{Hom}_{\text{sSet}}(\Delta^n, F).$$

Example A.1.6. 1. The total singular simplex $\text{Sing}_\bullet(X) = \{\text{Sing}_n(X)\}_n$ of a topological space X is a simplicial set.

2. ([Lur18a, Example 1.2.1.4]) Given a small category $\mathcal{C}$, we can associate a simplicial set, namely called the *nerve* of a category as follows:

   (a) $N\mathcal{C}_0$ are objects of $\mathcal{C}$.

   (b) $N\mathcal{C}_1$ are elements of the form $a \xrightarrow{f} b$ where $a, b \in \mathcal{C}$.

   (c) $N\mathcal{C}_2$ are 2-composable morphisms i.e. elements of the form

   $$a \xrightarrow{f} b \xrightarrow{g} c$$

   where $a, b, c \in \mathcal{C}$.

   (d) Inductively $N\mathcal{C}_n$ are n-composable morphisms i.e. elements of the form

   $$a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} a_3 \cdots \xrightarrow{f_{n-1}} a_n$$

   where $a_1, a_2, \ldots, a_n \in \mathcal{C}$.

(e) In context to the diagram of n-simplex, the face and degeneracy maps are defined as follows. The i^{th} face map maps a n-composable morphism of the form

$$a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} a_3 \cdots \xrightarrow{f_{n-1}} a_n$$

to an $(n-1)$-composable morphism

$$a_1 \xrightarrow{f_1} a_2 \cdots a_{i-1} \xrightarrow{f_i \circ f_{i-1}} a_{i+1} \cdots a_n.$$

The i th degeneracy maps a n-composable morphism

$$a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} a_3 \cdots \xrightarrow{f_{n-1}} a_n$$

to a $(n+1)$-composable morphism

$$a_1 \xrightarrow{f_1} a_2 \cdots a_i \xrightarrow{id_{a_i}} a_i \xrightarrow{f_{i+1}} a_{i+1} \cdots a_n.$$

A.2 ∞-categories: definition and basic terminologies.

At first, we recall the notion of ∞-groupoids or Kan complexes. Kan complexes are specific kind of simplicial sets which satisfy "extension along horns". At first, we define what do we mean by horns of the standard n-simplex.

Definition A.2.1. [Lur18a, Construction 1.1.2.9] The *kth horn* $|\Lambda^n_k|$ of $|\Delta^n|$ is the subcomplex of $|\Delta^n|$ obtained by removing the interior and the face opposite of the kth vertex. Denote Λ^n_k to be the associated simplicial set. Precisely, the m simplicies of the horn are:

$$\Lambda^n_k([m]) = \{\alpha \in \mathrm{Hom}_\Delta([m],[n]) | [n] \not\subset \alpha(m) \cup \{[k]\}\}$$

Remark A.2.2.
1. One should view Λ^n_k as removing the face opposite to the kth vertex in $\partial\Delta^n$.
2. We have obvious inclusion maps: $\Lambda^n_k \hookrightarrow \Delta^n$.

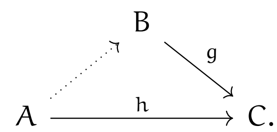
Definition A.2.3. [Lur18a, Definition 1.1.9.1] A simplicial set X satisfies the *Kan condition*, if given a morphism $\Lambda^n_k \to X$ can be extended to a map $\Delta^n \to X$, i.e. in other words there exists a dotted arrow such that the diagram commutes:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

Such simplicial sets are called *Kan complexes* or *fibrant* objects. More generally, a morphism $f : X \to Y$ of simplicial sets is called a *Kan fibration* if for all $0 \leq k \leq n$, a commutative diagram below, there exists a dotted arrow:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

If we closely look at the Kan condition it seems unreasonable to ask why the outer horns shall extend. For example in the case $n = 2$, it is unreasonable for the horns Λ^2_0, Λ^2_2 to extend. Visualizing this in diagram, these amounts to have the following dotted arrow in the diagram below (for the case Λ^0_2)



Thus extension of inner horns seems reasonable for arbitrary simplicial sets. These lead us to the definition of ∞-categories.

Definition A.2.4. [Lur18a, Definition 1.3.0.1] An ∞-*category* is a simplicial set K which has the following property: for any $0 < i < n$, every map $f_0 : \Lambda^n_i \to K$ of simplicial sets admits an extension $f : \Delta^n \to K$.

Example A.2.5. Here are some examples.

1. The nerve of any ordinary category $\mathcal{C}$ is an ∞-category. This because the extension from the inner horns to the whole simplex is uniquely defined by the composition of sequences of n maps.

2. It is clear from the definition that Kan complexes are ∞-categories.

A.2.1 Objects and morphisms in an ∞-category.

Definition A.2.6. [Lur18a, Definition 1.3.11] Let $\mathcal{C} = S_\bullet$ be an ∞-category.

1. An *object* of $\mathcal{C}$ is an element of S_0.

2. A *morphism* of $\mathcal{C}$ is an edge of the simplical set $S_\bullet$ i.e. an element of S_1.

3. Let $f \in S_1$ be a morphism in $\mathcal{C}$. We will refer $X = d_1(f)$ as the *source* of f and $d_0(f) = Y$ as the *target* of f.

4. For any object X, the degenerate edge $s_0(X)$ is a map from X to itself. We denote this morphism as the *identity* morphism of X, denoted by id_X.

Example A.2.7. 1. Let $N_\bullet(\mathcal{C})$ be the nerve of an ordinary category $\mathcal{C}$. Objects are objects of the ambient category $\mathcal{C}$. Morphisms are the ambient morphisms of $\mathcal{C}$.

2. For $\mathcal{C} = \mathrm{Sing}_\bullet(X)$, we have

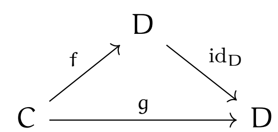
   (a) objects are points of the topological space X.

   (b) morphisms are paths in the topological space X.

A.2.2 Homotopies and composition of morphisms.

The concept of homotopies of morphisms allow us to define the homotopy category associated to an ∞-category. We refer to [Lur18a, Section 1.3.3 ,1.3.4,1,3,5] for more details.

Definition A.2.8. [Lur18a, Definition 1.3.3.1] Let $\mathcal{C}$ be an ∞-category and let $f, g : C \to D$ be a pair of morphisms in $\mathcal{C}$ having the same source and target. A *homotopy* from f to g is a 2-simplex σ of $\mathcal{C}$ satisfying $d_0(\sigma) = \mathrm{id}_D, d_1(\sigma) = g, d_2(\sigma) = f$, as depicted in the diagram



We will say f is *homotopic* to g if there exists a homotopy from f to g.

Theorem A.2.9. *[Lur18a, Proposition 1.3.3.5] Let X, Y be two objects of an ∞-category $\mathcal{C}$. Let E denote the collection of all morphisms from X to Y in $\mathcal{C}$. Then homotopy is an equivalence relation in E.*

Before defining homotopy classes, we need to know to define composition of morphisms in our ∞-category.

Definition A.2.10. [Lur18a, Definition 1.3.4.1] Let $\mathcal{C}$ be an ∞-category. Suppose we are given objects $X, Y \in \mathcal{C}$ and morphisms $f : X \to Y, g : Y \to Z$ and $h : X \to Z$. We will say that h is a *composition* of f and g if there exists a 2-simplex σ of $\mathcal{C}$ satisfying $d_0(\sigma) = g,\ d_1(\sigma) = h$ and $d_2(\sigma) = f$. In this case, we will also say that the 2-simplex σ *witnesses* h *as a composition of* f *and* g.

Remark A.2.11. Composition of morphisms exist by the property of an ∞-category. The above definition is independent of choice of morphisms upto homotopy. This is clear from the next proposition.

Proposition A.2.12. *[Lur18a, Proposition 1.3.4.2] Let $\mathcal{C}$ be an ∞-category containing morphisms $f : X \to Y$ and $g : Y \to Z$, then*

1. *Let $h : X \to Z$ which is a composition of f and g. Let $h' : X \to Z$ be another morphism with the same source and target. Then h' is a composition of f and g iff h' is homotopic to h.*

2. *Suppose $f' : X \to Y, g' : Y \to Z$ are morphisms homotopic to f and g respectively. Let h, h' be compositions of f, g and f', g' respectively. Then h' is homotopic to h.*

We are ready to define homotopy classes of morphisms.

Construction A.2.13. [Lur18a, Construction 1.3.5.1] Let $\mathcal{C}$ be an ∞-category. For every pair of objects $X, Y \in \mathcal{C}$, we let $\mathrm{Hom}_{h\mathcal{C}}(X, Y)$ denote the set of *homotopy classes* of morphisms from $X \to Y$ in $\mathcal{C}$. For every morphism $f : X \to Y$, we let $[f]$ denote its equivalence class in $\mathrm{Hom}_{h\mathcal{C}}(X, Y)$.
For morphisms classes $[f] \in \mathrm{Hom}_{h\mathcal{C}}(X, Y), [g] \in \mathrm{Hom}_{h\mathcal{C}}(Y, Z)$, we define the composition:

$$\circ : \mathrm{Hom}_{h\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{h\mathcal{C}}(X, Y) \to \mathrm{Hom}_{h\mathcal{C}}(X, Z)$$

as $[g] \circ [f] = [h]$ where h is a composition of f and g in the ∞-category $\mathcal{C}$.

Proposition A.2.14. *[Lur18a, Proposition 1.3.5.2] Let $\mathcal{C}$ be an ∞-category. Then:*

1. *The composition law defined above is associative.*

2. *The homotopy class $[\mathrm{id}_X]$ is a two sided identity with respect to the composition.*

We are ready to define the homotopy category.

Definition A.2.15. [Lur18a, Definition 1.3.5.3] Let $\mathcal{C}$ be an ∞-category, We define the category $h\mathcal{C}$ as follows:

- The objects of $h\mathcal{C}$ are objects of $\mathcal{C}$.

1. For every pair of objects $X, Y \in \mathcal{C}$, we let $\mathrm{Hom}_{h\mathcal{C}}(X, Y)$ denote the collection of homotopy classes of morphisms from X to Y (as in the Construction above).

2. Composition of morphisms are defined in the construction.

We call $h\mathcal{C}$ as the *homotopy category* of $\mathcal{C}$.

Remark A.2.16. The previous proposition gives us the fact that $h\mathcal{C}$ is a category.

A.2.3 Joins of ∞-categories.

At first, let us recall joins in classical category theory.

Definition A.2.17. [Lur09, Section 1.2.8] Let $\mathcal{C}, \mathcal{C}'$ be categories. We define the join of $\mathcal{C}$ and $\mathcal{C}'$, denoted by $\mathcal{C} * \mathcal{C}'$ as follows:

1. Objects: $\mathrm{ob}(\mathcal{C})$ or $\mathrm{ob}(\mathcal{C}')$.

2.
$$\mathrm{Hom}_{\mathcal{C}*\mathcal{C}'}(X, Y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X, Y) & X, Y \in \mathcal{C} \\ \mathrm{Hom}_{\mathcal{C}'}(X, Y) & X, Y \in \mathcal{C}' \\ \phi \quad X \in \mathcal{C}', Y \in \mathcal{C} \\ * \quad X \in \mathcal{C}, Y \in \mathcal{C}' \end{cases}$$

We now define joins for simplicial sets which will be joins for ∞-categories.

Definition A.2.18. [Lur09, Definition 1.2.8.1] Let S, S' be simplicial sets. We define the *join* of S, S', denoted by $S * S'$ as

$$(S * S')_n = S_n \cup S'_n \cup \bigcup_{i+j=n-1} S_i \times S'_j$$

Example A.2.19. We have $\Delta^j * \Delta^i \cong \Delta^{i+j-1}$. Also for ordinary categories $\mathcal{C}, \mathcal{C}'$, we have $N(\mathcal{C} * \mathcal{C}') \cong N(\mathcal{C}) * N(\mathcal{C}')$.

We have the following proposition.

Proposition A.2.20. *[Lur09, Proposition 1.2.8.3] Let S, S' be ∞-categories. Then the join $S * S'$ is also an ∞-category.*

Proof. Let $p : \Lambda^n_i \to S * S'$ be the map. If p maps entirely into S or S', then we have the extension by the property of S, S' being ∞-categories.
Now suppose p maps vertices $0, 1, ..j$ to S and $j+1, j+2, ...n$ to S', by extension property of S and S' we have the maps:

$$p_j : \Delta^{\{0,1,2..j\}} \to S, p'_j : \Delta^{\{j+1,j+2,..n\}} \to S'$$

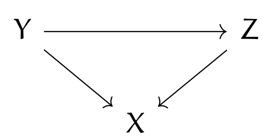
Together we get a map $\Delta^n \to S * S'$. □

We shall use the following useful notation:

Notation A.2.21. [Lur09, Notation 1.2.8.4] Let K be a simplicial set. Then $K^{\triangleleft}$ is called the *left cone* defined by $\Delta^0 * K$. Dually we call $K^{\triangleright}$ to be the *right cone* which is defined by $K * \Delta^0$.

A.2.4 Over and undercategories of an ∞-category.

In classical category theory, we have the concept of over and undercategories. Given a category $\mathcal{C}$ and an object $x \in \mathcal{C}$, we have the overcategory defined as:

1. Objects:$Y \to X$ morphisms in $\mathcal{C}$.
2. Morphisms: 

Choosing an object X is a map of simplicial sets $X : \Delta^0 \to N(\mathcal{C})$. The analogue of objects and morphisms of overcategory for $N(\mathcal{C})$ can be defined as follows:

1. Objects $p_0 : \Delta^0 * \Delta^0 \to N(\mathcal{C})$ where the restriction on second factor gives X.
2. Morphisms $p_1 : \Delta^1 * \Delta^0 \to N(\mathcal{C})$ where restricting on second factor gives X.

This motivates us to give the following definition of overcategory for simplicial sets and thus ∞-categories.

Definition A.2.22. [Lur09, Proposition 1.2.9.2, Remark 1.2.9.5] Given $p : K \to S$ an aribtrary map. There exists a simplicial set $S_{/p}$ which is defined as follows:

$$(S_{/p})_n = \mathrm{Hom}_p(\Delta^n * K, S)$$

where the subscript on the right hand side indicates that we consider only morphisms $f : \Delta^n * K \to S$ such that $f|_K = p$. $S_{/p}$ is called the *overcategory* associated to p.

Dually, we can define the undercategory with respect to a map of simplicial sets.

Remark A.2.23. [Lur09, Remark 1.2.9.6] Let $\mathcal{C}$ be an ordinary category. Let X be an object of $\mathcal{C}$ (also an object of $N(\mathcal{C})$). We have the following equivalence $N(\mathcal{C})_{/X} \cong N(\mathcal{C}_{/X})$.

A.2.5 Initial and final objects of an ∞-category.

Let X be an object in an ordinary category $\mathcal{C}$. Then X is said to be final if for all objects $Y \in \mathcal{C}$, we have an unique morphism $f_Y : Y \to X$. X is said to be initial if it is final in the category $\mathcal{C}^{op}$.

In the language of ∞-categories, we define such notions in the following manner.

Definition A.2.24. [Lur09, Definition 1.2.12.3] Let $\mathcal{C}$ be an ∞-category, a vertex $x \in \mathcal{C}$ (i.e. $x \in \mathcal{C}_0$) is said to be *final/ initial* if for any map $f : \partial\Delta^n \to \mathcal{C}$ such that $f([n]) = x(f([0]) = x)$, it extends to a morphism $\tilde{f} : \Delta^n \to \mathcal{C}$.

Remark A.2.25. In the case of $n = 1$, we see the condition in the case of classical category theory. In fact, it can be see that the for an object $y \in \mathcal{C}$ and if x is initial/final object of the category, the mapping space $\mathrm{Hom}_R(x, y)$ $(\mathrm{Hom}_R(y, x))$ is contractible.

A.2.6 Diagrams, limits and colimits in an ∞-category.

In classical category theory, a diagram is a morphism from an indexing category to the category. In the simplicial world, it is defined as follows:

Definition A.2.26. A diagram in a simplicial set C is a morphism of simplicial sets $F : K \to C$.

A limit of a diagram in the context of classical category theory is defined via the universal property. An elegant way to view this is to see in the following manner:
Let $\mathcal{C}$ be a category and I be an indexing category. Let $F : I \to \mathcal{C}$ be a functor Consider the overcategory $\mathcal{C}$ over F denoted as $\mathcal{C}_F$. This is defined as:

1. Objects: $\{\text{Diagrams} : \gamma_{C,i} : F(C) \to F(I)\}_{C \in \mathcal{C}}$ with usual commutativity relations.
2. Morphisms: Morphisms between diagrams with commutativity conditions.

Now a limit of the functor F is the final object in the category $\mathcal{C}_F$. The usual universal property translates into the condition of the object being final in this category.

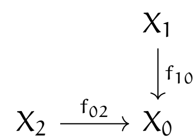
We use this equivalent definition in the ∞-categorical setting.

Definition A.2.27. [Lur09, Definition 1.2.13.4] Let $\mathcal{C}$ be an ∞-category. Let $f : K \to \mathcal{C}$ be an arbitrary map of simplicial sets. Then a **limit** of f is a final object in the overcategory $\mathcal{C}_{/f}$. Dually, a **colimit** of F is an initial object in the undercategory $\mathcal{C}_{f/}$.

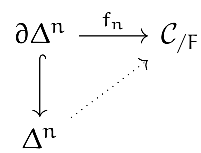
Remark A.2.28. Let us spell out the definition in a special case. Let I be the three object category considered as a simplicial set:

$$\begin{array}{ccc} & & i_1 \\ & & \downarrow \\ i_2 & \longrightarrow & i_0 \end{array}$$

Let $F : I \to \mathcal{C}$ be a functor where $\mathcal{C}$ is an ∞-category. Denote the diagram as:

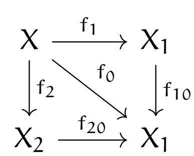


We want to see what the universal property of the limit of F is. Consider the overcategory $\mathcal{C}_{/F}$. Let X be the limit of F. Thus given any map $f_n : \partial\Delta^n \to \mathcal{C}_{/F}$ with $f_n([n]) = x$, there exists a map $\tilde{f_n} : \Delta^n \to \mathcal{C}_{/F}$ such that the diagram



commutes. Let us spell out the above condition case by case what this means:

1. **n=0**: In this case, the condition just says that we have a unique morphism $\Delta^0 \to \mathcal{C}_{/F}$, i.e. a map $\Delta^0 * I \to \mathcal{C}$ such that when restricted to I it is F. Thus we have a following diagram:



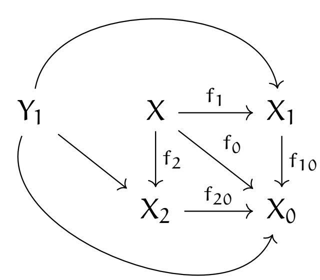
   along with homotopies on the triangles which are 2-simplexes in the ambient category $\mathcal{C}$.

2. **n=1**: In this case, we are given a map $\partial\Delta^1 \to \mathcal{C}_{/F}$ which sends $[1] \to X$, thus we have a morphism of simplicial sets $\partial\Delta^1 * I \to \mathcal{C}$ such that:

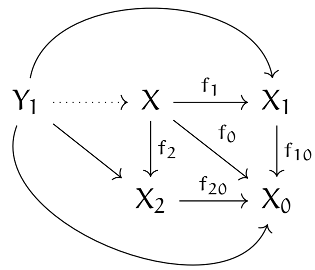
   (a) [1] should be sent to X.

   (b) When restricted to I, it should be F.

   Unravelling this morphism as a diagram, we get the following:



with additional homotopies on the triangles $Y_1X_2X_0$ and $Y_1X_1X_0$.
The existence of the morphism $\Delta^1 \to \mathcal{C}_{/F}$ now translates to existence of a dotted arrow in the diagram:



such that we have 2-simplices: Y_1XX_1, YXX_2 and YXX_0 and 3-simplices: YXX_1X_0 and YXX_2X_0.

Notice this is exactly a generalization of the universal property of limits in classical category theory.

3. For cases $n = 2$ and higher cases, we need to replace Y_1 and X_1 by a boundary of an n-simplex where the nth coordinate is X. The existence of the unique morphism says that we can fill the boundary to get an n-simplex and additional homotopies in the diagram which are compatible with one another.

A.3 Relation between simplicial and ∞-categories.

This section recalls the connection between simplicial and ∞-categories. The main reference is [Lur09, Section 1.1.4,1.1.5]

Definition A.3.1. [Lur09, Definition 1.1.4.1] A *simplicial category* is a category which is enriched over category of simplicial sets Set_Δ.

Simplicial categories are quite common, the category sSet is a simplicial category. We have similar notions of homotopy categories of simplicial categories like ∞-categories. Thus it is reasonable to expect a connection between infinity categories and simplicial categories. The goal of this section is to define the "**homotopy coherent nerve**" of a simplicial category. This gives a formal notion of **homotopy coherent diagrams** which is a core idea for this whole language.

Recall, that given a small category $\mathcal{C}$, we have $N(\mathcal{C})$ the usual nerve. This gives us a functor

$$N : \mathrm{Cat} \to \mathrm{sSet}$$

More precisely, we defined the simplicial set $N(\mathcal{C})$ as:

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, N(\mathcal{C})) = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$$

where $[n]$ is the linearly ordered set considered as a category. Our goal is to define

$$N_{sm} : \mathrm{Cat}_\Delta \to \mathrm{sSet}$$

We expect it to be defined in the similar way as the usual nerve, but we need to encode the simplicial structure in the definition. Thus we need to replace $[n]$ by a "thickening" to get a simplicial category $\mathfrak{C}[\Delta^n]$. Our first goal is to define such simplicial categories.

Definition A.3.2. [Lur09, Definition 1.1.5.1] Let J be a finite nonempty linearly ordered set. The simplicial category $\mathfrak{C}[\Delta^J]$ is defined as follows:

1. Objects: Elements of J.
2. If $i, j \in J$, then
$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i,j) = \begin{cases} \phi & \text{if } j < i \\ N(P_{ij}) & \text{if } i \leq j \end{cases}$$
Here P_{ij} denotes the partially ordered set $\{I \subset J : (i,j) \in I \wedge (\forall k \in I)[i \leq k \leq j]\}$

If $i_0 \leq i_1 \leq i_2. \leq \cdots \leq i_n$, then the composition

$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0,i_1) \times \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_1,i_2) \times \cdots \times \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_{n-1},i_n) \to \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0,i_n)$$

is induced by the map of partially ordered sets:

$$P_{i_0 i_1} \times P_{i_1 i_2} \times \cdots \times P_{i_{n-1} i_n} \to P_{i_0 i_n}$$

defined as

$$(I_1, I_2, \cdots, I_n) \to I_1 \cup I_2 \cup \cdots \cup I_n$$

Remark A.3.3. Let us try to understand the definition in special cases. Let $J = [2]$ be the linearly ordered set. Then the category $\mathfrak{C}[\Delta^2] := \mathfrak{C}[\Delta^J]$ has objects $0, 1$ and 2. Denote p_{ij} to be the unique morphism from i to j when $i \leq j$.
We have the following morphisms:

1. $\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(0,1) = N(\{\{0,1\}\})$.
2. $\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(1,2) = N(\{\{1,2\ \}\})$.
3. $\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(0,2) = N(\{\{0,2\}\{0,1,2\}\})$.

Let us denote q_{ij} to be the partially ordered set $\{i,j\}$. Thus we can see that by the definition of $\mathfrak{C}[\Delta^2]$, we denote $\{0,1,2\}$ as $q_{12} \circ q_{01}$.
1' Notice that $p_{02} = p_{12} \circ p_{01}$ in [2], whereas in $\mathfrak{C}[\Delta^2]$, $q_{02} \neq q_{12} \circ q_{01}$. But we notice that q_{02} is homotopic to $q_{12} \circ q_{01}$ in $\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(0,2)$. Thus $\mathfrak{C}[\Delta^2]$ is the category 2, where we have dropped the associativity condition

$$p_{02} = p_{12} \circ p_{01}$$

but the composition is homotopic to p_{02}.
There is a similar explanation for $\mathfrak{C}[\Delta^n]$, it is the category $[n]$ where we have dropped the associativity conditions

$$p_{ik} = p_{jk} \circ p_{ij} \ \ ; \ \ i < j < k$$

Instead, all sorts of compositions of the form

$$p_{i_0k} \circ p_{i_1i_0} \circ \cdots \circ p_{i_{n-1}i_n}$$

for $i < i_n < i_{n-1} < \cdots i_0 < k$ are homotopic to p_{ik}.

Definition A.3.4. [Lur09, Definition 1.1.5.5] Let $\mathcal{C}$ be a simplicial category. The *simplicial nerve* $N_{sm}(\mathcal{C})$ is described by the formula

$$\mathrm{Hom}_{\mathrm{Sets}_\Delta}(\Delta^n, N_{sm}(\mathcal{C})) = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C}) \quad (A.1)$$

We have the following proposition.

Proposition A.3.5. *[Lur09, Proposition 1.1.5.10] Let $\mathcal{C}$ be a simplicial category such that for any two objects $X, Y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex. Then the simplicial nerve $N_{sm}(\mathcal{C})$ is an ∞-category.*

A.4 Mapping spaces in ∞-categories.

The Hom sets in ∞-categories are not only sets but they are objects in the homotopy category of spaces. In this subsection, we introduce two notions of mapping spaces which represent the same homotopy type.

Definition A.4.1. [Lur09, Definition 1.1.3.2, Example 1.1.3.3] Let Top be the category of CW complexes where for given X and Y two topological spaces, we have $\mathrm{Mor}_{\mathrm{Top}}(X, Y)$ to be space of continuous maps between topological spaces equipped with compact open topology. Thus Top is a topological category. We define to be the homotopy category of Top and we call it to be the *homotopy category of spaces.*

Definition A.4.2. Given a simplicial category C, we can associate a topological category $|C|$ as follows:

1. Objects are objects of C.
2. $\mathrm{Hom}_{|C|}(X, Y) = |\mathrm{Hom}_C(X, Y)|$.

Definition A.4.3. [Lur09, Definition 1.2.2.1] For S, a simplicial set and $x, y \in S_0$, we define

$$\mathrm{Map}_S(x, y) = \mathrm{Hom}_{h|\mathfrak{C}[S]|}(x, y) \in \mathcal{H}.$$

Remark A.4.4. Some remarks on the definition of $\mathrm{Map}_S(x, y)$:

1. An advantage of this definition is that it can be defined for an arbitrary simplicial set.
2. A disadvantage of this definition is that it is not usually a Kan complex, hence homotopic theoretic computations become more difficult.

There is a second candidate for mapping spaces which is relevant in ∞-categorical setting.

Definition A.4.5. [Lur09, Section 1.2.2] Let S be a simplicial set with objects x, y, then we define the simplicial set $\mathsf{opHom}^R_S(x, y)$ as follows:

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{Hom}^R_S(x, y)) = \{z : \Delta^{n+1} \to S | z|_{\Delta^{n+1}} = y, z|_{\Delta^{0,\cdots,n}} = x\}$$

We have the following important proposition on the above definition

Proposition A.4.6. *[Lur09, Proposition 1.2.2.3] Let $\mathcal{C}$ be an ∞-category with objects x, y, then $\mathrm{Hom}^R_{\mathcal{C}}(x, y)$ is a Kan complex.*

A.5 Subcategories of ∞-categories.

The notion of subcategories in ∞-categories is induced from the level of homotopy categories.

Definition A.5.1. [Lur09, Section 1.2.11] Let $\mathcal{C}$ be an ∞-category. Let $(h\mathcal{C})' \subset h\mathcal{C}$ be a subcategory. Let $\mathcal{C}'$ be the pullback

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ N(h\mathcal{C})' & \longrightarrow & N(h\mathcal{C}) \end{array} \tag{A.2}$$

in Set_Δ. We say $\mathcal{C}'$ is the *subcategory of* $\mathcal{C}$ *spanned by* $(h\mathcal{C})'$. Also $\mathcal{C}' \subset \mathcal{C}$ is a subcategory if it arises in this fashion.
Morever we say $\mathcal{C}'$ is full if $(h\mathcal{C})'$ is full subcategory of $h\mathcal{C}$.

A.6 Relation between 2-categories and ∞-categories.

As we have seen before, given any ordinary small category, the nerve construction gives you an ∞-category. What about 2-categories? The reason for why we are actually concerned about this is because we shall work with the 2-category of algebraic stacks. At first, we recall the definition of 2-categories, followed by the Duskin Nerve which yields an infinity category from a special class of 2-categories. The main tool for defining the motivic homotopy theory of algebraic stacks requires an ∞-category associated to 2-category of algebraic stacks. The main reference is [Lur18a, Section 2.2,2.3]

A.6.1 2-Categories.

Definition A.6.1. [Lur09, Definition 2.2.1.1] A *2-category* $\mathcal{C}$ consists of the following data:

1. A collection $\mathrm{Ob}(\mathcal{C})$ whose elements we refer to as *objects of* $\mathcal{C}$. We will often abuse notation by writing $X \in \mathcal{C}$ to indicate that X is an element of $\mathrm{Ob}(\mathcal{C})$.

2. For every pair of objects $X, Y \in \mathcal{C}$, a category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$. We will refer to objects of the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ as *1-morphisms* from X to Y. We denote those objects as $f : X \to Y$. Given a pair of 1-morphisms f, g in $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$, we refer to morphisms between f and g as *2-morphisms* from f to g. We will denote them as $\gamma : f \Rightarrow g$.

3. For every triple of objects X, Y, Z in $\mathcal{C}$, a *composition functor*
$$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z)$$

4. For every object $X \in \mathcal{C}$, we have a 1-morphism $\mathrm{id}_X \in \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$, which we call the identity 1-morphism from X to itself.

5. For every object $X \in \mathcal{C}$, an isomorphism $\mu_X : \mathrm{id}_X \circ \mathrm{id}_X \Rightarrow \mathrm{id}_X$ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$. We refer to the 2-morphisms $\{\mu_X\}_{X \in \mathcal{C}}$ as the *unit constraints* of $\mathcal{C}$.

6. For every quadrauple objects $W, X, Y, Z \in \mathcal{C}$, a natural isomorphism α from the functor

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(W, X) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(W.Z) \quad (h, g, f) \to h \circ (g \circ f)$$

to the functor

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(W, X) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(W.Z) \quad (h, g, f) \to (h \circ g) \circ f.$$

We denote the value on a triple (h, g, f) by $\alpha_{h,g,f} : h \circ (g \circ f) \Rightarrow (h \circ g) \circ f$. We refer to these isomorphisms as the *associativity constraints* of $\mathcal{C}$.

7. For every pair of objects $X, Y \in \mathcal{C}$, the functors

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \quad f \to f \circ \mathrm{id}_X$$

and

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \quad f \to \mathrm{id}_Y \circ f$$

are fully faithful.

8. For every quadruple of composable 1-morphisms, there is a commutative diagram in shape of a pentagon.

Definition A.6.2. [Lur18a, Definition 2.2.0.1] Let $\mathcal{C}$ be a 2-category. Then $\mathcal{C}$ is said to be a *strict 2-category* if the unit and the associativity constraints are the identity 2-morphisms in $\mathcal{C}$.

We have another kind of constraints in 2 categories which can be deduced from the definition.

Definition A.6.3. [Lur18a, Construction 2.2.1.12] Let $\mathcal{C}$ be a 2-category. For every 1-morphism $f : X \to Y$ in $\mathcal{C}$, we have canonical ismorphisms

$$\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \xrightarrow{\alpha_{\mathrm{id}_Y, \mathrm{id}_Y, f}} (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \xrightarrow{\mu_Y \circ \mathrm{id}_f} \mathrm{id}_Y \circ f$$

As the comoposition on the left with identity is fully faithful, it follows there exists a unique isomorphism $\lambda_f : \mathrm{id}_Y \circ f \Rightarrow f$ for which the diagram

$$\begin{array}{ccc} \mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) & \xRightarrow{\alpha_{\mathrm{id}_Y, \mathrm{id}_Y, f}} & (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \\ & {\scriptstyle \mathrm{id}_{\mathrm{id}_Y} \circ \lambda_f} \searrow \quad \swarrow {\scriptstyle \mu_Y \circ \mathrm{id}_f} & \\ & \mathrm{id}_Y \circ f & \end{array}$$

commutes We will refer λ_f as the *left unit constraint*. Similarly, we have $\rho_f : f \circ \mathrm{id}_X \Rightarrow f$ called the *right unit constraint* for which a similar kind of diagram exists like above which commutes.

We now define lax functors between 2-categories.

Definition A.6.4. [Lur18a, Definition 2.2.4.5] Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. A *lax functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

1. For every object X in $\mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.

2. For every pair of objects $X, Y \in \mathcal{C}$, a functor of ordinary categories:

$$F_{X,Y} : \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(Y))$$

   We will denote $F(f)$ when $F_{X,Y}$ is evaluated on an object f in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$. Also, we shall denote $F(\gamma)$ when $F_{X,Y}$ is evaluated at a morphism γ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$.

3. For every morphism $X \in \mathcal{C}$, a 2-morphism $\epsilon_X : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X)$ in the 2-category $\mathcal{D}$, which we will refer to as the *identity constraint*.

4. For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the 2-category $\mathcal{C}$, A 2-morphism

$$\mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$$

   which we will refer to as the *composition constraint*. We also require that if the objects X, Y and Z are fixed, then the construction $(g, f) \to \mu_{g,f}$ is functorial.

These datas are required to satisfy compatibility with associativity and the unit constraints.

Definition A.6.5. [Lur18a, Definition 2.2.4.5] A *functor* from $\mathcal{C}$ to $\mathcal{D}$ is a lax functor $F : \mathcal{C} \to \mathcal{D}$ with the property that the identity and composition constraints are isomorphisms.

Definition A.6.6. [Lur18a, Definition 2.2.4.17] Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a lax functor. We say that F is *unitary* if for every object $X \in \mathcal{C}$, the identity constraints ϵ_X is an invertible 2-morphism of $\mathcal{D}$. We also say that F is *strictly unitary* if for every object $X \in \mathcal{C}$, we have an equality $\mathrm{id}_{F(X)} = F(\mathrm{id}_X)$ and the identity constraint ϵ_X is the identity 2-morphism from $\mathrm{id}_{F(X)}$ to itself.

A.6.2 The Duskin nerve.

In this subsection, we describe how to construct a simplicial set from a 2-category. This is done by the Duskin Nerve. Before that we make a definition on a special class of 2-categories we are interested in.

Definition A.6.7. [Lur18a, Construction 2.3.0.1] A *(2,1)-category* is a 2-category $\mathcal{C}$ with the property that every 2-morphism in $\mathcal{C}$ is invertible.

Example A.6.8. We define the 2-category of algebraic stacks as:

1. Objects: Algebraic stacks.

2. 1-morphisms: Morphisms of algebraic stacks.

3. 2-morphisms: Natural transformations which are isomorphisms.

Thus the 2-category of algebraic stacks is a $(2, 1)$-category.

Definition A.6.9. [Lur18a, Definition 2.3.1.1] Let n be a non-negative integer and let $[n]$ be the usual linearly ordered set. We will regard $[n]$ as a category and as a 2-category where 2-morphisms are the identity 2-morphisms. For any 2-category $\mathcal{C}$, we denote

$$\mathsf{N}^{\mathsf{D}}_n(\mathcal{C}) = \{\text{strictly unitary lax functors from } [n] \to \mathcal{C}\}$$

This determines

$$\mathsf{N}^{\mathsf{D}}_{\bullet}(\mathcal{C}) : \Delta^{op} \to \text{Sets}\,.$$

We will denote $\mathsf{N}^{\mathsf{D}}_{\bullet}(\mathcal{C})$ as the *Duskin Nerve* of $\mathcal{C}$.

We have the following theorem which relates $(2,1)$-categories with ∞-categories.

Theorem A.6.10. *[Lur18a, Theorem 2.3.2.1] Let $\mathcal{C}$ be a 2-category. Then $\mathcal{C}$ is a $(2,1)$-category iff $\mathsf{N}^{\mathsf{D}}_{\bullet}(\mathcal{C})$ is an ∞-category.*

A.7 The ∞-category of spaces.

In this section, we define the ∞-categorical analogue of Sets.

Definition A.7.1. [Lur09, Definition 1.2.16.1] Let $\mathcal{K}an$ denote the full subcategory of Set_{Δ} spanned by (small) Kan complexes. We define $\mathcal{S}$ to be the simplicial nerve of $\mathcal{K}an$. We refer $\mathcal{S}$ to be the *∞-category of spaces.*

Let $\widehat{\mathcal{S}}$ be the simplicial nerve of simplicial category spanned by large Kan complexes.

Proposition A.7.2. *[Lur09, Remark 1.2.16.2] The simplicial sets $\widehat{\mathcal{S}}$ and $\mathcal{S}$ are ∞-categories.*

A.8 Model structure and fibrations of simplicial sets.

In this section, we start by recalling fibrations of simplicial sets. Fibrations play an important role in defining model structure on the category of simplicial sets. They play an important role in phrasing the Grothendieck construction of fibered categories in the ∞-categorical setting.

A.8.1 Recalling notions of fibrations on simplicial sets.

We recall the various notions of fibrations and anodyne in the category of simplicial sets.
An important notion for these definitions is the notion of left lifting and right lifting property which is used in the context of model categories.

A morphism $p : X \to Y$ of simplicial sets has *right lifting property(RLP)* with respect to $q : Z \to W$ if for a commutative diagram of the form

$$\begin{CD} Z @>>> X \\ @VqVV @VVpV \\ W @>>> Y \end{CD}$$

there exists a dotted arrow $r : W \to X$ such that both of the triangles commutes.
We have the dual notion of *left lifting property (LLP)*.

Definition A.8.1. [Lur09, Definition 2.0.0.3] Let $f : X \to S$ be a morphism of simplicial sets.

1. f is a *Kan fibration* if it has RLP with respect to $\Lambda_i^n \subset \Delta^n$ for $0 \le i \le n$ for all $n \ge 0$.
2. f is a *trivial fibration* if it has LLP with respect to $\partial\Delta^n \subset \Delta^n$ for all $n \ge 0$.
3. f is a *left (right) fibration* if it has RLP with respect to $\Lambda_i^n \subset \Delta^n$ for $0 \le i < n (0 < i \le n)$.
4. f is an *inner fibration* if it has RLP with respect to $\Lambda_i^n \subset \Delta^n$ for $0 < i < n$.
5. f is *left (right) anodyne* if it has LLP with respect to left(right) fibrations.
6. f is *inner anodyne* if it has LLP with respect to inner fibrations.
7. f is a *categorical equivalence* if for any other ∞-category B, the map induced by f

$$Y^B \to X^B$$

is a equivalence on the level of homotopy categories. Here $Y^B := \mathrm{Fun}(Y, B)$.

Lemma A.8.2. *[Rez, Section 21] Inner anodynes and trivial fibrations are categorical equivalences.*

A.8.2 Model structures on simplicial sets.

We recall two model structures on the category of simplicial sets Sets_Δ,namely the Quillen Model structure and Joyal Model structure. We refer to [Lur09, Chapter 2, Appendix A.2]

Definition A.8.3. We define the Model structure on Sets_Δ, denoted by $\mathrm{Sets}_{\Delta,Q}$ as follows:

1. Cofibrations are level wise injections of simplicial sets.
2. Weak equivalences are weak homotopy equivalences i.e. morphisms whose geometric realization is a weak homotopy equivalence of topological spaces.
3. Fibrations are Kan fibrations.

Remark A.8.4. In $\mathrm{Set}_{\Delta,Q}$, the Kan complexes are fibrant objects.

Definition A.8.5. We have the Joyal model structure on Set_Δ, denoted by $\mathrm{Set}_{\Delta,J}$ defined by the following class of morphisms:

1. Cofibrations are levelwise injections.
2. Weak equivalences are *categorical equivalences*.
3. Fibrations are inner fibrations.

Remark A.8.6. The ∞-categories are fibrant objects in $\mathrm{Set}_{\Delta,J}$

A.8.3 Cartesian fibrations.

Cartesian fibrations are ∞-categorical generalizations of category fibered over categories. At first, we need to have an ∞-categorical generalization of cartesian morphisms.

p-Cartesian morphisms.

Definition A.8.7. [Lur09, Definition 2.4.1.1] $p : X \to S$ an inner fibration of simplicial sets. Let $f : x \to y$ be an edge in X. We shall say that f is p-*Cartesian* if the induced map

$$X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$$

is a trivial Kan fibration.

Remark A.8.8. Let us get the explanation of the definition step by step. Recall that $\Delta^m * \Delta^n \cong \Delta^{m+n+1}$.

<u>**Condition of trivial fibration at level of objects.**</u>

1. $(X_{/f})_0 = \begin{array}{ccc} z & \longrightarrow & x \\ & \searrow & \swarrow_{f} \\ & y & \end{array}$,

2. $(X_{/y})_0 = (m \to y)$.

3. $(S_{/p(f)})_0 = \begin{array}{ccc} m' & \longrightarrow & p(x) \\ & \searrow & \swarrow_{p(f)} \\ & p(y) & \end{array}$,

Thus the map at the level 0 is the usual map. Being a Kan fibration, means that given $z \to y$ a morphism such that $p(x), p(y)$ and $p(z)$ form a 2-simplex, then there exists a map $z \to y$ such that z, x, y form a 2-simplex.

This shows the analog of a morphism being Cartesian in category theory.

.

Proposition A.8.9. *[Lur09, Remark 2.4.1.4] Let $p : X \to S$ be an inner fibration. Then $f : \Delta^1 \to X$ is p-Cartesian iff we have the following dotted arrow:*

$$\begin{array}{ccc} \Delta^{\{n-1,n\}} & & \\ \downarrow & \searrow^{f} & \\ \Lambda^n_n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & S \end{array}$$

for $n \geq 2$.

We now recall the definition of cartesian fibrations.

Definition A.8.10. [Lur09, Definition 2.4.2.1] A map $p : X \to S$ is a *Cartesian fibration* if the following conditions hold

1. p is an inner fibration.

2. Let $f : x \to y$ an edge in S and $\tilde{y}$ a vertex of X such that $p(\tilde{y}) = y$, then there exists a p-Cartesian edge $\tilde{f} : \tilde{x} \to \tilde{y}$ such that $p(\tilde{f}) = f$.

Dually, we have the notion of coCartesian fibrations just taking the opposite of the maps of simplicial sets being Cartesian fibrations.

A.8.4 Isofibrations.

Isofibrations are morphisms of simplicial sets which lift equivalences. We state the definition and some properties of isofibrations. The main reference of this section is [Lan21, Section 2.1].

Definition A.8.11. [Lan21, Definition 2.1.4] A inner fibration $p : \mathcal{C} \to \mathcal{D}$ is called an *isofibration* if every lifting problem

$$\begin{array}{ccc} \{0\} & \longrightarrow & \mathcal{C} \\ \big\downarrow & \nearrow & \big\downarrow p \\ \Delta^1 & \xrightarrow{f} & \mathcal{D} \end{array} \tag{A.3}$$

where f represents an equivalence of $\mathcal{D}$ has a solution which represents an equivalence of $\mathcal{C}$.

We list some properties of isofibrations.

Lemma A.8.12. *[Lan21, Corollay 2.1.6, Lemma 2.1.7]*

1. *Left and right fibrations between ∞-categories are isofibrations.*

2. *p is an isofibration iff p^{op} is.*

3. *For any ∞-category $\mathcal{D}$ and $n \geq 1$, the restriction map $\mathrm{Fun}(\Delta^n, \mathcal{D}) \to \mathrm{Fun}(\partial\Delta^n, \mathcal{D})$ is an isofibration.*

Remark A.8.13. The third property in the above lemma is a special case of [Lan21, Proposition 2.2.5].

Lemma A.8.14. *Let $\mathcal{D}$ be an ∞-category and $n \geq 1$. Let*

$$f : \Delta^1 \times \partial\Delta^n \coprod_{\{0\}\times\partial\Delta^n} \{0\} \times \Delta^n \to \mathcal{D}$$

be a morphism such that $f|_{\Delta^1\times[k]} : \Delta^1 \to \mathcal{D}$ is an equivalence for all $0 \leq ke \leq n$. Then there exists a morphism $f' : \Delta^n \times \Delta^1 \to \mathcal{D}$ such that the diagram

$$\begin{array}{ccc} \Delta^1 \times \partial\Delta^n \coprod_{\{0\}\times\partial\Delta^n} \{0\} \times \Delta^n & \xrightarrow{f} & \mathcal{D} \\ \big\downarrow & \nearrow_{f'} & \\ \Delta^n \times \Delta^1 & & \end{array} \tag{A.4}$$

commutes.

Proof. The morphism f gives us the following commutative diagram

$$\begin{array}{ccc} \{0\} & \longrightarrow & \mathrm{Fun}(\Delta^n, \mathcal{D}) \\ \big\downarrow & & \big\downarrow p \\ \Delta^1 & \xrightarrow{g} & \mathrm{Fun}(\partial\Delta^n, \mathcal{D}) \end{array} \tag{A.5}$$

where g is an equivalence. By the previous lemma, the map p is an isofibration. Thus there exists a morphism $f' : \Delta^1 \times \mathrm{Fun}(\Delta^n, \mathcal{D})$ which extends f. This completes the proof. ☐

A.9 The category of ∞-categories.

In this section, we define the ∞-category of ∞-categories Cat_∞.

Definition A.9.1. $\widehat{\mathrm{Cat}_\infty^\Delta}(\mathrm{Cat}_\infty^\Delta)$ is the simplicial category which is defined as follows:

1. Objects: (small) ∞-categories.

2. Morphisms: $\mathrm{Map}_{\mathrm{Cat}_\infty^\Delta}(\mathcal{C}, \mathcal{D}) =$ Largest Kan complex inside $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

We define $\mathrm{Cat}_\infty := \mathsf{N}_{sm}(\mathrm{Cat}_\infty^\Delta)$ and $\widehat{\mathrm{Cat}_\infty} := \mathsf{N}_{sm}(\widehat{\mathrm{Cat}_\infty^\Delta})$ (where N_{sm} is the simplicial nerve).

A.10 Cofinal maps.

In this section, we briefly recall the notion of cofinal maps. Given a functor $f : K \to K'$ of simplicial sets, then sometimes $\mathrm{colim}(f)$ can be computed by a simpler diagram instead of K. The theory of cofinal maps generalizes this formalism. We recall the definition of cofinal maps ([Lur09, Section 4.1]) and state some important properties of such maps. An important application of cofinal maps is that limits of simplicial objects and semi-simplicial objects are the same.

Definition A.10.1. [Lur09, Definiton 4.1.1.1] Let $p : S \to T$ be a map of simplicial sets. We say that p is *cofinal* if for any right fibration $X \to T$, the induced map of simplicial sets

$$\mathrm{Map}_T(T, X) \to \mathrm{Map}_T(S, X)$$

is a homotopy equivalence.

Remark A.10.2. There is a dual notion of final maps where we replace right by left fibrations.

Proposition A.10.3. *[Lur09, Proposition 4.1.1.3, 4.1.1.8]*

1. *An isomorphism of simplicial sets is cofinal.*

2. *An inclusion of simplicial sets is cofinal iff it is a right anodyne.*

3. *A map of simplicial sets $\nu : K' \to K$ is cofinal iff for any diagram $\bar{p} : K^{\triangleright} \to \mathcal{C}$ which is a colimit of $p = \bar{p}|_K$, the induced map $\bar{p}' : K'^{\triangleright} \to \mathcal{C}$ is a colimit of $p' = \bar{p}'|_{K'}$.*

An important application of cofinality is the following corollary.

Corollary A.10.4. *[Lur09, Corollay 5.1.2.3] Let K and S be simplicial sets. Let $\mathcal{C}$ be an ∞-category such that $\mathcal{C}$ admits K-indexed limits and colimits. The ∞-category $\mathrm{Fun}(\mathsf{S}, \mathcal{C})$ admits K-indexed limits and colimits. In particular, the ∞-category of presheaves over S denoted by $\mathcal{P}(\mathsf{S})$ admits all limits and colimits.*

Now we state an important application of cofinality on limits on simplicial and semisimplicial objects.

Definition A.10.5. Let Δ_s to be the sub-simplicial set of Δ where the objects are same as Δ but the morphisms are only injective maps (i.e. spanned by face maps).

Lemma A.10.6. *[Lur09, Lemma 6.5.3.7] The inclusion $i : \mathsf{N}(\Delta_s^{op}) \to \mathsf{N}(\Delta)^{op}$ is cofinal, i.e. taking colimits in $\widehat{\mathrm{Cat}_\infty}$ indexed by $\mathsf{N}(\Delta_s^{op})$ and $\mathsf{N}(\Delta^{op})$ are same.*

A.11 Kan extensions.

Let $\mathcal{C}$ and $\mathcal{J}$ be ordinary categories. We have the diagonal functor:

$$\delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$$

The diagonal functor has a left adjoint if $\mathcal{C}$ admits small colimits. The left adjoint is described as:

$$\mathcal{C}^{\mathcal{J}} \to \mathcal{C} : f \to \operatorname{colim} f$$

This allows us to colimits as left adjoints to diagonal functors.
More generally, if we have a map $i : \mathcal{J} \to \mathcal{J}'$ a map between diagram categories,then we have the induced functor:

$$i^* : \mathcal{C}^{\mathcal{J}'} \to \mathcal{C}^{\mathcal{J}}.$$

If $\mathcal{C}$ has sufficient supply of colimits, one can construct a left adjoint to i^*. This is called as the *left Kan extension along* i.
In this section, we recall the notion of Kan extensions along inclusion of ∞-categories. The main reference for this section is [Lur09, Section 4.3.2]. In order to recall the precise definition, we recall the notion of relative colimits.

A.11.1 Relative colimits.

Relative colimits are generalization of usual colimits, but along inner fibrations. The definition is as follows:

Definition A.11.1. [Lur09, Definition 4.3.1.1] Let $f : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets, let $\bar{p} : \mathsf{K}^{\triangleright} \to \mathcal{C}$ be a diagram and let $p = \bar{p}|_{\mathsf{K}}$. We will say $\bar{p}$ is a f-*colimit of* p if the map:

$$\mathcal{C}_{\bar{p}/} \to \mathcal{C}_{p/} \times_{\mathcal{D}_{fp/}} \mathcal{D}_{f\bar{p}/}$$

is a trivial fibration of simplicial sets. In the case, we will also say that $\bar{p}$ is a f-*colimit diagram*.

Remark A.11.2. When $f : \mathcal{C} \to \Delta^0$ is the morphism where $\mathcal{C}$ is an ∞-category, then the notion of f-colimit is same as that of colimit. In this sense, this notion is a generalization of the usual notion of colimits.

A.11.2 Kan extensions along inclusions.

Notation A.11.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Let $\mathcal{C}^0$ be a full subcategory of $\mathcal{C}$. If C is an object of $\mathcal{C}$, let $\mathcal{C}^0_{/C}$ be the full subcategory of $\mathcal{C}_{/C}$ spanned by the morphisms $C' \to C$ where $C' \in \mathcal{C}^0$.

Definition A.11.4. [Lur09, Definition 4.3.2.2] Suppose we are given a commutative diagram of ∞-categories:

$$\begin{array}{ccc} \mathcal{C}^0 & \xrightarrow{F_0} & \mathcal{D} \\ \big\downarrow & \overset{F}{\nearrow} & \big\downarrow p \\ \mathcal{C} & \longrightarrow & \mathcal{D}' \end{array}$$

where:

1. p is an inner fibration.
2. The left vertical map is an inclusion of a full subcategory.

We will say F is a p-*left Kan extension of* F_0 *at* $C \in \mathcal{C}$ if the induced diagram:

$$\begin{array}{ccc} (\mathcal{C}^0_{/C}) & \xrightarrow{F_C} & \mathcal{D} \\ \big\downarrow & \nearrow & \big\downarrow p \\ (\mathcal{C}^{(0)}_{/C})^{\triangleright} & \longrightarrow & \mathcal{D}' \end{array}$$

exhibits $F(C)$ as a p-colimit of F_C.

We will say that F is a p-*left Kan extension of* F_0 if it is a p-left Kan extension of F_0 at every $C \in \mathcal{C}$.

Also if $\mathcal{D}' = \Delta^0$, then we just say that F is a *left Kan extension of* F_0 if the above conditions are satisfied.

Remark A.11.5. The property of left Kan extensions in a naive sense can be formulated in the following way: Given a diagram:

$$\begin{array}{ccc} \mathcal{C}^0 & \xrightarrow{F_0} & \mathcal{D} \\ \big\downarrow & \overset{F}{\nearrow} & \\ \mathcal{C} & & \end{array}$$

Then F is a left Kan extension of F_0 if for every $C \in \mathcal{C}$, $F(C) \cong \lim_{C^0 \in \mathcal{C}^{(0)}_C} F(C^0)$. Thus a Kan extension of F_0 on the level of objects is just taking colimits on the overcategory $\mathcal{C}^{(0)}_C$.

In the case when $\mathcal{C} = \mathcal{C}^{(0)\triangleright}$, the condition of left Kan extensions is equivalent to say that F is a p-colimit of F_0.

The following proposition is an ∞-categorical generalization of the fact that a morphism between directed (inverse) systems induce a morphism between colimits (limits).

Proposition A.11.6. *(Simplified version of [Lur09, Corollary 4.3.2.16]) Let $\mathcal{C}$ and $\mathcal{D}$ be two ∞-categories. Let $i : \mathcal{C}^{(0)} \subset \mathcal{C}$ be a full subcategory of $\mathcal{C}$. Suppose that for every functor $F_0 \in \mathrm{Fun}(\mathcal{C}^{(0)}, \mathcal{D})$, there exists a functor $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ which is a left Kan extension of F_0. Then the restriction map $i^* : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^{(0)}, \mathcal{D})$ induced by i admits a section $i_! : \mathrm{Fun}(\mathcal{C}^{(0)}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose essential image consists of precisely those functors F which are left Kan extensions of $F|_{\mathcal{C}^{(0)}}$.*

Remark A.11.7. Let $\mathcal{C}^{(0)} = K$ and $\mathcal{C} = K^{\triangleright}$. The condition of left Kan extension is equivalent to the condition of colimit (Remark A.11.5). Thus the map $i_!$ on the level of objects sends a functor $f : K \to \mathcal{D}$ to its colimit diagram $\tilde{f} : K^{\triangleright} \to \mathcal{D}$.
There is a dual notion of the proposition in terms of right Kan extensions. In the dual setting, one gets a functorial assignment of limit diagrams.

Another application of Kan extensions is the following proposition.

Proposition A.11.8. *[Lur09, Lemma 5.5.2.3] Let $H : X^{\triangleleft} \times Y^{\triangleleft} \to \mathcal{D}$ where $\mathcal{D}$ is an infinity category. Assume the following:*

1. *For every $x \in X^{\triangleleft}$, $H_x : Y^{\triangleleft} \to \mathcal{D}$ is a colimit diagram.*
2. *For every $y \in Y$, $H_y : X^{\triangleleft} \to \mathcal{D}$ is a colimit diagram.*

Then let ∞ be the cone point of $Y^{\triangleleft}$, then $H_\infty : X^{\triangleleft} \to \mathcal{D}$ is a colimit diagram.

Remark A.11.9.
1. The above theorem can be formulated in the following naive way: Assume that:
   (a) $\mathrm{colim}_x H(x, y)$ for $y \in Y$ exists
   (b) $\mathrm{colim}_y H(x, y)$ exists
   (c) $\mathrm{colim}_x \mathrm{colim}_y H(x, y)$ exists.

   Then $\mathrm{colim}_y \mathrm{colim}_x H(x, y)$ exists and $\mathrm{colim}_x \mathrm{colim}_y H(x, y) \cong \mathrm{colim}_x \mathrm{colim}_y H(x, y)$.

A.12 The ∞-categorical Yoneda lemma.

In this section, we recall the notion of presheaves and state the Yoneda Lemma.

Definition A.12.1. [Lur09, Definition 5.1.0.1] Let S be a simplicial set. We denote $\mathcal{P}(S) := \mathrm{Fun}(S^{op}, \mathcal{S})$ and call it as ∞-*category of presheaves on* S.

We recall the construction of the Yoneda functor:

Construction A.12.2. [Lur09, Section 5.1.3] Let K be a simplicial set.

1. Let $\mathcal{C} = \mathfrak{C}[K]$.We have a natural map

$$a_K : \mathfrak{C}[K^{op} \times K] \to \mathcal{C}^{op} \times \mathcal{C}$$

   by the properties of the fiber product and the functor $\mathfrak{C}$.

2. We have a simplicial functor
$$b_K : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{K}an$$
given by
$$(X, Y) \to \mathrm{Sing}\,|\mathrm{Hom}_{\mathcal{C}}(X, Y)|$$

3. We have the composition of simplicial functors:
$$b_K \circ a_K : \mathfrak{C}[K^{op} \times K] \to \mathcal{K}an$$

4. By adjunction of $\mathfrak{C}$ and N_{sm}, we get a map of simplicial sets:
$$K^{op} \times K \to \mathcal{S}$$

5. This gives us the Yoneda embedding:
$$j : K \to \mathcal{P}(K)$$

Proposition A.12.3. *[Lur09, Proposition 5.1.3.1] Let K be a simplicial set. Then the Yoneda Embedding j is fully faithful.*

An important fact of Yoneda lemma is the fact that $\mathcal{P}(K)$ for a simplicial set K is "freely generated" by K under small colimits. We make this more precise below.

Notation A.12.4. [Lur09, Notation 5.1.5.1] Let $\mathcal{C}, \mathcal{D}$ be ∞-categories. Denote $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ be the subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by colimit-preserving functors.

Definition A.12.5. [Lur09, Definition 5.1.5.7] Let $\mathcal{C}$ be an ∞-category. A subcategory $\mathcal{C}' \subset \mathcal{C}$ is said to be *stable under colimits* if for any small diagram $p : K \to \mathcal{C}'$ which has a colimit $p' : K^{\triangleright} \to \mathcal{C}$, the map p' factors through $\mathcal{C}'$.
Let A be a collection of objects in $\mathcal{C}$, then A *generates* $\mathcal{C}$ under colimits if for any subcategory $\mathcal{C}''$ of $\mathcal{C}$ containing A is stable under colimits, then $\mathcal{C}'' \equiv \mathcal{C}$. A map of simplicial set $f : S \to \mathcal{C}$ *generates* $\mathcal{C}$ under small colimits if the image $f(S_0)$ generates $\mathcal{C}$ under small colimits.

The above notations help us to recall the theorem which gives a condition when a functor $f : S \to \mathcal{C}$ extends to a functor $\tilde{f} : \mathcal{P}(S) \to \mathcal{C}$.

Theorem A.12.6. *[Lur09, Theorem 5.1.5.6] Let S be a simplicial set and $\mathcal{C}$ be an ∞-category admitting all small colimits. Then the functor $j : S \to \mathcal{P}(S)$ induces an equivalence of ∞-categories*
$$\mathrm{Fun}^L(\mathcal{P}(S), \mathcal{C}) \to \mathrm{Fun}(S, \mathcal{C})$$

Corollary A.12.7. *[Lur09, Corollary 5.1.5.8] Let S be a simplicial set. Then the Yoneda embedding $j : S \to \mathcal{P}(S)$ generates S under small colimits.*

A.13 Adjoint functors.

In this section, we recall the notion of adjoint functors. One can define adjoint functors in terms of correspondences (see [Lur09, Section 5.1, 5.2] for more details). We give an equivalent definition of adjoint functors in terms of unit transformation. Recall that in classical category theory, the relationship between a pair of adjoint functors is given by specifying a *unit transformation*. This concept can be generalized to ∞-categorical setting as follows:

Definition A.13.1. [Lur09, Definition 5.2.2.7] Suppose given a pair of functors

$$f : \mathcal{C} \leftrightarrows \mathcal{D} : g$$

between ∞-categories. A *unit transformation* for (f, g) is a morphism $u : \mathrm{id}_{\mathcal{C}} \to g \circ f$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ with the following property: for every pair of objects $C \in \mathcal{C}, D \in \mathcal{D}$, the composition

$$\mathrm{Map}_{\mathcal{D}}(f(C), D) \to \mathrm{Map}_{\mathcal{C}}(g(f(C)), g(D)) \xrightarrow{u(C)} \mathrm{Map}_{\mathcal{C}}(C, g(D))$$

is an isomorphism in the homotopy category $\mathcal{H}$.

Definition A.13.2. [Lur09, Proposition 5.2.2.8] Let $f : \mathcal{C} \to \mathcal{D}, g : \mathcal{D} \to \mathcal{C}$ be a pair of functors between ∞-categories $\mathcal{C}$ and $\mathcal{D}$. Then f is *left adjoint* to g if there exists a unit transformation $u : \mathrm{id}_{\mathcal{C}} \to g \circ f$.

Next we recall the notion of adjointable squares and ∞-category of left-adjointable functors defined in [Lur17, Section 4.7] which help us to state the base change theorems in six functor formalism.

Definition A.13.3. [Lur17, 4.7.4.13] Suppose we have a diagram of ∞-categories

$$\sigma := \begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle V} \\ \mathcal{C}' & \xrightarrow{G'} & \mathcal{D}' \end{array} \tag{A.6}$$

which commutes upto specified equivalence

$$\alpha : V \circ G \cong G' \circ U.$$

We say σ is *left adjointable* if G and G' admit left adjoints F and F' respectively and if the composite transformation

$$F' \circ V \to F' \circ V \circ G \circ F \cong F' \circ G' \circ U \circ F \to U \circ F$$

is an equivalence.

Remark A.13.4. Some remarks on the definition above:

1. We have the dual notion of *right adjointable squares*.
2. The notion of left and right adjointable squares in classical category theory is called the *Beck-Chevalley condition*.

Definition A.13.5. [Lur17, Definition 4.7.4.16] Let S be a simplicial set, we define subcategories

$$\mathrm{Fun}^{\mathsf{LAd}}(S, \mathrm{Cat}_\infty), \mathrm{Fun}^{\mathsf{RAd}}(S, \mathrm{Cat}_\infty) \subset \mathrm{Fun}(S, \mathrm{Cat}_\infty)$$

as follows:

1. Let $F \in \mathrm{Fun}(S, \mathrm{Cat}_\infty)$. Then $F \in \mathrm{Fun}^{\mathsf{LAd}}(S, \mathrm{Cat}_\infty)(\mathrm{Fun}^{\mathsf{RAd}}(S, \mathrm{Cat}_\infty))$ if and only if for every edge $s \in s'$ $F(s) \to F(s')$ admits a left (right) adjoint.

2. Let $\alpha : F \to F'$ be a morphism in $\mathrm{Fun}(S, \mathrm{Cat}_\infty)$ where $F, F' \in \mathrm{Fun}^{\mathsf{LAd}}(S, \mathrm{Cat}_\infty)$ $(\mathrm{Fun}^{\mathsf{RAd}}(S, \mathrm{Cat}_\infty))$, then α is a morphism in $\mathrm{Fun}^{\mathsf{LAd}}(S, \mathrm{Cat}_\infty)(\mathrm{Fun}^{\mathsf{RAd}}(S, \mathrm{Cat}_\infty))$ if for every $s \in s'$, the diagram

$$\begin{array}{ccc} F(s) & \longrightarrow & F(s') \\ \downarrow & & \downarrow \\ F'(s) & \longrightarrow & F'(s') \end{array} \tag{A.7}$$

is left(right) adjointable.

We have the following important corollary.

Corollary A.13.6. *[Lur09, Corollary 4.7.4.18]*

1. *The ∞-categories* $\mathrm{Fun}^{\mathsf{LAd}}(S, \mathrm{Cat}_\infty)$ *and* $\mathrm{Fun}^{\mathsf{RAd}}(S, \mathrm{Cat}_\infty)$ *are presentable and in particular, they admit all small limits.*

2. *There is a canonical equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathsf{LAd}}(S^{op}, \mathrm{Cat}_\infty) \cong \mathrm{Fun}^{\mathsf{RAd}}(S, \mathrm{Cat}_\infty).$$

A.14 Localization of ∞-categories.

In classical category, we have the notion of localization of a category. A typical example of a localization is the construction of derived category which is the localization of the category of chain complexes under quasi-isomorphisms. We briefly recall the notion of localization in the context of ∞-categories. The references for this section are [Lan21, Section 2.4] and [Lur18a, Section 6.3].

Definition A.14.1. Let $\mathcal{C}$ be an ∞-category and let $S \subset \mathcal{C}_1$ be a subset of morphisms in $\mathcal{C}$. A functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ is called a *Dwyer-Kan localization* of $\mathcal{C}$ along S if for every ∞-category $\mathcal{D}$, the restriction functor

$$\mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful and the essential image consists of those functors that send S to equivalences.

Let J be the contractible Kan complex consisting of two objects. Then a typical example of a localization is the following.

Lemma A.14.2. *The anodyne map* $e : \Delta^1 \to J$ *is localization along the unique morphism* $0 \to 1$.

We list some properties of localization.

Proposition A.14.3.
1. *If a localization exists, then it is uniquely determined upto categorical equivalence.*
2. *For every* $S \subset \mathcal{C}_1$*, there exists a localization of* $\mathcal{C}$ *along* S.

Remark A.14.4. The existence of the localization is shown by a functorial construction of the localization of $\mathcal{C}$ along a collection of edges S. At first, we define the simplicial set $\mathcal{C}'$ as the pushout of the diagram

$$\begin{array}{ccc} \coprod_{s\in S}\Delta^1 & \longrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle \coprod e} & & \downarrow{\scriptstyle e'} \\ \coprod_{s\in S} J & \longrightarrow & \mathcal{C}'. \end{array} \tag{A.8}$$

Note that $\mathcal{C}'$ is a simplicial set in general and the morphism e' is an anodyne. By small object argument, there exists a functorial factorization of the morphism $\mathcal{C}' \to \Delta^0$ as

$$\mathcal{C}' \xrightarrow{i'} \mathcal{C}[S^{-1}] \to \Delta^0$$

where i' is an inner anodyne and $\mathcal{C}[S^{-1}]$ is an ∞-category (see [Lur18a, Proposition 4.2.3.1]). The composition $i : \mathcal{C} \xrightarrow{e'} \mathcal{C}' \xrightarrow{i'} \mathcal{C}[S^{-1}]$ is indeed the localization of $\mathcal{C}$ along S. The objects of $\mathcal{C}[S^{-1}]$ by construction are same as objects of $\mathcal{C}$. On the level of edges, we have added inverses of morphisms in S.

A.15 Presentable ∞-categories.

In this section, we give a brief recall of the notion of presentable ∞-categories from [Lur09, Section 5.3-5.5]. Presentable ∞-categories are special kind of subcategories of ∞-categories of presheaves which admit "small" conditions in order to ensure good supply of limits and colimits. The adjoint functor theorem is formulated in the setting of presentable ∞-categories.

In order to define the notion of presentability, we need to recall the notion of Ind-objects in the context of ∞-categories. Recall that in classical category theory, Ind-objects are direct limits over filtered categories. Thus we recall the notion of filtered ∞-categories. In order to keep track of set-theoretical issues, we need the concept of regular cardinal.

Notation A.15.1. Regular cardinal are such cardinals which are cannot be decomposed into subsets of smaller cardinal. We shall denote the cardinal $\aleph_0$ by ω. Regular cardinals shall be denote by κ.

Definition A.15.2. [Lur09, Definition 5.3.1.7] Let κ be a regular cardinal and let $\mathcal{C}$ be an ∞-category. We say $\mathcal{C}$ is κ-*filtered* if for every small simplicial set K and every map $f : K \to \mathcal{C}$, there exists a map $\bar{f} : K^{\triangleright} \to \mathcal{C}$ extending f. $\mathcal{C}$ is said to be *filtered* if $\mathcal{C}$ is ω filtered.

Example A.15.3. If $\mathcal{C} = \mathsf{N}(A)$ where A is a partially ordered set. Then $\mathcal{C}$ is κ-filtered if and only if every κ-small subset of A has an upper bound in A.

We now move to recall the notion of Ind-objects in the context of ∞-categories. Before recalling the definition, let us briefly recall another description of Ind-objects in the context of classical category theory. Given any ordinary category C, the category of Ind-objects is the subcategory of presheaves over C spanned by objects which are functors $F : C^{op} \to \mathrm{Sets}$ such that

$$F \cong \mathrm{colim}_{d \in D}\, j(\alpha(D))$$

where $\alpha : D \to C$ is a functor from a filtered category D and j is yoneda embedding. The ∞-categorical definition is the generalization of this above description.

Definition A.15.4. [Lur09, Corollary 5.3.5.4] Let $\mathcal{C}$ be a small ∞-category and κ be a regular cardinal. Then the ∞-category of κ- Ind-*objects* over $\mathcal{C}$ is the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by functors $F : \mathcal{C}^{op} \to \mathcal{S}$ such that F is a colimit of the composition

$$j \circ p : \mathcal{J} \to \mathcal{P}(\mathcal{C})$$

where $p : \mathcal{J} \to \mathcal{C}$ is a functor from a κ-filtered ∞-category $\mathcal{J}$ and j is the ∞-categorical Yoneda embedding. We shall denote the ∞-category of κ-Ind-objects by $\mathrm{Ind}_\kappa(\mathcal{C})$.

Remark A.15.5. The Yoneda embedding j factors through $\mathrm{Ind}_\kappa(\mathcal{C})$.

We are ready to recall the definition of presentable ∞-category in terms of Ind-objects.

Definition A.15.6. An ∞ category is *presentable* if there exist a regular cardinal κ and a small ∞-category $\mathcal{D}$ such that there is an equivalence $\mathrm{Ind}_\kappa(\mathcal{D}) \to \mathcal{C}$.

Example A.15.7. The ∞ category $\mathcal{S}$ is presentable.

Now we look at the case of representable functors and the adjoint functor theorem. Let $F : \mathcal{C}^{op} \to \mathcal{S}$ be a functor which is representable (i.e. it lies in the essential image of the Yoneda Embedding). It turns out that F preserves limits. This is because we have the decomposition

$$\mathcal{C}^{op} \xrightarrow{j} \mathcal{P}(\mathcal{C}^{op}) \xrightarrow{\mathrm{ev}_C} \mathcal{S}$$

which is a composition of preserving limit functors. It turns out that the converse holds when $\mathcal{C}$ is presentable.

Theorem A.15.8. *Let $\mathcal{C}$ be a presentable ∞ category and let $F : \mathcal{C}^{op} \to \mathcal{S}$ be a functor. The following are equivalent:*

1. *F is representable by an object $C \in \mathcal{C}$.*
2. *The functor F preserves small limits.*

Another application of presentable ∞-categories is the adjoint functor theorem which gives criterion of the existence of left and right adjoints in the setting of presentable ∞-categories.

Theorem A.15.9 (Adjoint functor theorem)**.** *[Lur09, Corollary 5.5.2.9] Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories. Then*

1. *The functor* F *has a right adjoint iff it is accessible and preserves small colimits.*

2. *The functor* F *has a left adjoint iff it is accessible and preserves small limits.*

Remark A.15.10. We did not recall the definition of a functor being accessible. In the context of presentable ∞-categories, a functor is accessible if it preserved κ-filtered colimits. For the precise definition, see [Lur09, Definition 5.4.2.5].

Next, we define recall the definition of subcategories of $\widehat{\mathrm{Cat}_\infty}$ which contain presentable ∞-categories.

Notation A.15.11. [Lur09, Definition 5.5.3.1] Let $\widehat{\mathrm{Cat}_\infty}$ be the category of ∞-categories. We define two subcategories $\mathrm{Pr}^L, \mathrm{Pr}^R \subset \widehat{\mathrm{Cat}_\infty}$ as follows:

1. The objects in $\mathrm{Pr}^L, \mathrm{Pr}^R$ are presentable ∞-categories.

2. A morphism in Pr^L are functors between presentable ∞-categories which preserve small colimits.

3. A morphism in Pr^R are functors between presentable ∞-categories which are accessible and preserve limits.

Proposition A.15.12. *[Lur09, Proposition 5.5.3.13, Theorem 5.5.3.18] The ∞-categories $\mathrm{Pr}^L(\mathrm{Pr}^R)$ admits all small limits(colimits).*

A.16 Abstract descent theory.

In this section, we recall the notion of abstract descent theory. Descent theory is an important notion of defining sheaves in classical algebraic geometry. One of the main advantages of ∞-categories is that descent theory works really well. More precisely, presheaves like Derived categories, K-theory are not sheaves in the classical setting. The language of ∞-categories allows us to treat them as ∞-sheaves.
At first, we start by recalling the notion of Čech nerve of a morphism.

Definition A.16.1. [Lur09, Proposition 6.1.2.6] Let $\mathcal{C}$ be an ∞-category and $U : N(\Delta)^{op} \to \mathcal{C}$ be a simplicial object of $\mathcal{C}$. Denote $U_n := U([n])$. Then U is called a groupoid object if for every $n \geq 0$ and for every partition $[n] = S \cup S'$ such that $S \cap S' = \{s\}$, then the diagram

$$\begin{array}{ccc} U_n & \longrightarrow & U(S) \\ \downarrow & & \downarrow \\ U(S') & \longrightarrow & U(\{s\}) \end{array}$$

is a pullback diagram in $\mathcal{C}$.

Notation A.16.2. Let $\Delta_+^{\leq n}$ denoted the full subcategory of Δ_+ spanned by objects $[k]_{-1\leq k\leq n}$.

Definition A.16.3. [Lur09, Proposition 6.1.2.11] An augmented simplicial object $U_\bullet^+ : N(\Delta_s)^{op} \to \mathcal{C}$ is a *Čech nerve* if the augmented simplicial object $U_\bullet^+$ is a right Kan extension of

$$U_\bullet^+|_{N(\Delta_+^{\leq 0})^{op}}.$$

Remark A.16.4. In the case above, $U_\bullet^+$ is determined by the map $u : U_0 \to U_{-1}$ upto equivalence. Thus, we will say $U_\bullet^+$ is the *Čech nerve of* u.

Notation A.16.5. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism in $\mathcal{C}$, then we shall denote the Čech nerve of f by $Y^+_{\bullet,f}$.

Definition A.16.6. [Lur09, Remark 6.1.2.13] Let $U_\bullet$ be a simplicial object of an ∞-category $\mathcal{C}$. Denote $|U_\bullet| : N(\Delta_+)^{op} \to \mathcal{C}$ be a colimit of $U_\bullet$. We will refer to any such colimit as *geometric realization* of $U_\bullet$.

We define the notion of F-descent.

Definition A.16.7. [LZ17, Defintion 3.1.1] Let $\mathcal{C}$ be an ∞-category which admits pullbacks. $F : \mathcal{C}^{op} \to \mathcal{D}$ be a functor of ∞-categories and $f : X_0^+ \to X_1^+$ be a morphism in $\mathcal{C}$. Then f satisfies F-*descent* if

$$F \circ (X_\bullet^+)^{op} : N(\Delta_+) \to \mathcal{D}$$

is a limit diagram where $X_\bullet^+$ is the Čech nerve of f.

The notion of F-descent helps us to state the following lemma.

Lemma A.16.8. *Let $\mathcal{D}$ be an ∞-category which admits products. Let*

$$\begin{array}{ccc} D_3 & \xrightarrow{f_{32}} & D_2 \\ \downarrow{\scriptstyle f_{31}} & & \downarrow{\scriptstyle f_{20}} \\ D_1 & \xrightarrow{f_{10}} & D_0 \end{array}$$

be a commutative diagram in $\mathcal{C}$. Let $F : \mathcal{D}^{op} \to \mathcal{D}'$ be a functor of infinity categories. Assume the following:

1. *f_{32} and f_{31} satisfies F-descent.*
2. *f_{20} satisfies F-descent.*

Then f_{10} satisfies F-descent. Also, we have:

$$\lim_{n\in\Delta^{op}} F(D_{1n}) \xrightarrow{\cong} \lim_{n\in\Delta^{op}} F(D_{3n}) \xleftarrow{\cong} \lim_{m\in\Delta^{op}} F(D_{2m}) \tag{A.9}$$

where $D_{1n} := D_1^{\times n}$ over D_0,$D_{2m} := D_2^{\times m}$ over D_0 and $D_{3n} := D_{1n} \times_{D_0} D_{2n}$.

Proof. We define $H' : N(\Delta_+)^{op} \times N(\Delta_+)^{op} \to \mathcal{D}$ be the functor induced by the taking iterated fiber products of the commutative square. Define $H = F \circ H'$ and apply Proposition A.11.8 to the functor H. The assumptions in the corollary are the conditions of the colimit conditions in Proposition A.11.8.
Also notice that D_0 is the limit of the $H|_{N(\Delta)^{op}\times N(\Delta)^{op}}$. As the simplicial set $N(\Delta)^{op}$ is sifted ([Lur09, Lemma 5.5.8.4]), the diagonal map is cofinal. This implies Eq. (A.9). □

A.17 Sheaves on ∞-categories.

This section recalls the notion of sheaves in ∞-categorical setting. At first, we recall the notion of sieves.

Definition A.17.1. [Lur09, Definition 6.2.2.1] Let $\mathcal{C}$ be an ∞-category. A sieve on $\mathcal{C}$ is a fullsubcategory $\mathcal{C}^{(0)} \subset \mathcal{C}$ having the property that if $f : C \to D$ is a morphism in $\mathcal{C}$ and D belongs to $\mathcal{C}^{(0)}$, then C belongs to $\mathcal{C}^{(0)}$.
If $C \in \mathcal{C}^{(0)}$, then a sieve on C is a sieve on the ∞-category $\mathcal{C}_{/C}$.

Definition A.17.2. A Grothendieck topology on an ∞-category $\mathcal{C}$ consists of a specification, for each object C of $\mathcal{C}$, of a collection of sieves which we will refer to as *covering sieves*. A sieve is a covering sieve are required to possess the following properties:

1. If C is an object of $\mathcal{C}$, then the sieve $\mathcal{C}_{/C}$ is a covering sieve on C.

2. If $f : C \to D$ is a morphism in $\mathcal{C}$ and $\mathcal{C}^{(0)}_{/C}$ is a covering sieve on D, then $f^*\mathcal{C}^{(0)}_{/C}$ is a covering sieve on C.

3. Let C be an object of $\mathcal{C}$, $\mathcal{C}^{(0)}_{/C}$ is a covering sieve on C and $\mathcal{C}^{(1)}_{/C}$ is an arbitrary sieve. Suppose for each $f : D \to C$ belonging to the sieve $\mathcal{C}^{(0)}_{/C}$,the pullback $f^*\mathcal{C}^{(1)}_{/C}$ is a covering sieve on D. Then $\mathcal{C}^{(1)}_{/C}$ is a covering sieve on C.

Remark A.17.3. 1. If $\mathcal{C}$ is the nerve of ordinary category, then the definition reduces to the usual notion of a Grothendieck topology on $\mathcal{C}$.

2. For a small ∞-category, we have the following bijection:

$$\{\text{Monomorphisms } U \to j(C)\} \leftrightarrow \{\text{Covering sieves on } C\}$$

for any $C \in \mathcal{C}$ where j is the Yoneda embedding ([Lur09, Proposition 6.2.2.5]).

Definition A.17.4. [Lur09, Definition 6.2.2.6] Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology. Let S be the collection of all monomorphism $U \to j(C)$ which correspond to covering sieves over C. An object $\mathcal{F} \in \mathcal{P}(\mathcal{C})$ is a *sheaf* if it is S-local i.e. for every $\mathcal{G} \in \mathcal{P}(\mathcal{C})$, we have an isomorphism of topological spaces

$$\text{Map}_{\mathcal{C}}(j(C), \mathcal{G}) \to \text{Map}_{\mathcal{C}}(U, \mathcal{G})$$

for all monomorphism $U \to j(C)$ where $U \in \mathcal{P}(\mathcal{C})$.

In general, we can define for any arbitrary ∞-category $\mathcal{D}$, a *$\mathcal{D}$-valued sheaf* as follows;

Definition A.17.5. Let $\mathcal{D}$ be any arbitrary ∞-category and $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{D}$ be a functor. Let $\mathcal{C}$ be euqipped with a Grothendieck topology, then $\mathcal{F}$ is said to be a *$\mathcal{D}$-valued sheaf* if for all $C \in \mathcal{C}$ and for every covering sieve $\mathcal{C}^{(0)}_{C}$, we have

$$\mathcal{F}(C) = \lim \mathcal{F}|_{\mathcal{C}^{(0)\,op}_{C}}.$$

Generally, we are given a collection of morphisms and it is useful to know when it is possible to construct a Grothendieck topology generated by these collection of morphisms called as *covering morphisms.*The following proposition gives the possibility of conditions to get a Grothendieck topology.

Proposition A.17.6. *[Lur18b, Prop A.3.2.1] Let $\mathcal{C}$ be an ∞-category and S be a collection of morphisms in $\mathcal{C}$. Assume that:*

1. *The collection of morphisms S contains all equivalences and is stable under composition.*
2. *The ∞-category $\mathcal{C}$ admits pullbacks and S is stable under pullbacks.*
3. *The ∞-category $\mathcal{C}$ admits finite coproducts and S admits finite coproducts.*
4. *Finite coproducts in $\mathcal{C}$ are universal. That is, given the pullback diagram:*

$$\begin{array}{ccc} \coprod_{i\le i\le n} C_i \times_D D' & \longrightarrow & D' \\ \downarrow & & \downarrow \\ \coprod_{1\le i\le n} C_i & \longrightarrow & D, \end{array}$$

   we have

$$\coprod_{1\le i\le n} C_i \times_{D'} D \coprod \coprod_{1\le i\le n} (C_i \times_{D'} D).$$

Then there exists a Grothendieck topology on $\mathcal{C}$ defined as follows: A sieve $\mathcal{C}_C^{(0)} \subset \mathcal{C}_C$ over any object C is a covering sieve if it contains a finite collection of morphisms $\{C_i \to C\}_{1\le i\le m}$ such that $\coprod_{1\le i\le m} C_i \to C$ is in S.

The next proposition gives us an equivalent condition when sheaf condition is same as using descent along Čech covers.

Proposition A.17.7. *[Lur18b, Prop A.3.3.1] Let $\mathcal{C}$ be an ∞-category and S be a collection of morphisms. Assume that $\mathcal{C}$ and S satisfy conditions of Proposition A.17.6 together with the following additional hypothesis:*

1. *Coproducts in $\mathcal{C}$ are disjoint, i.e., let $*$ the initial object of $\mathcal{C}$, then the cocartesian diagram*

$$\begin{array}{ccc} * & \longrightarrow & C \\ \downarrow & & \downarrow \\ C' & \longrightarrow & C' \coprod C \end{array}$$

   is cartesian.

Let $\mathcal{D}$ be an arbitrary category and $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{D}$ be a functor. Then $\mathcal{F}$ is a $\mathcal{D}$-valued sheaf iff:

1. *$\mathcal{F}$ preserves finite products.*

2. *Let $f : U_0 \to X$ be a morphism in $\mathcal{C}$ which lies in S. Let $U_\bullet$ be the Čech nerve associated to the morphism f, regarded as an augmented simplicial object of $\mathcal{C}$. Then the composite map*
$$N(\Delta_+)^{op} \xrightarrow{U_\bullet} \mathcal{C}^{op} \xrightarrow{\mathcal{F}} \mathcal{D}$$
is a limit diagram. In other words, $F(X)$ is the totalization of the cosimplicial object $[n] \to \mathcal{F}(U_n)$ where $U_n = U_X^{n+1}$.

The Grothendieck topologies that we consider in the categories that we are interested in will satisfy the conditions Proposition A.17.6 and Proposition A.17.7. Thus the sheaf condition reduces to checking descent along Čech covers.

A.18 Stable ∞-categories.

In this section, we recall the notion of stable ∞-categories. Stable ∞-categories incorporate the notions of triangulated category in the ∞-language. They also incorporate the notions of stable homotopy theory, namely fiber and cofiber sequences are same. One of the main important examples of stable ∞-categories are the ∞-derived category $\mathcal{D}(\mathcal{A})$ associated to an abelian category and the ∞-category of spectra Sp. We end the section studying properties of presentable stable ∞-categories.

We start by recalling some terminologies to define stable ∞-categories.

Definition A.18.1. [Lur17, Definition 1.1.1.1] Let $\mathcal{C}$ be an ∞-category. A *zero object* of $\mathcal{C}$ is both an initial and final object of $\mathcal{C}$. A category $\mathcal{C}$ is *pointed* if it contains a zero object.

Remark A.18.2. If C is an additive category with a zero object 0 in the classical sense, then 0 is zero object in the ∞-category $N(C)$ in the above definition.

We recall the notion of fiber and cofiber sequences.

Definition A.18.3. [Lur17, Definition 1.1.1.4] Let $\mathcal{C}$ be an ∞-category. A *triangle* in $\mathcal{C}$ is a square $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form
$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z. \end{array}$$
It is a *(co)fiber sequence* if the square is a (pushout)pullback.

Remark A.18.4. If $\mathcal{C}$ is the nerve of an ordinary category C, the square in the above definition is a fiber (cofiber) sequence is equivalent of saying that X (Z) is the kernel(cokernel) of the morphism $Y \to Z$ ($X \to Y$).

Definition A.18.5. [Lur17, Definition 1.1.1.6] Let $\mathcal{C}$ be a pointed ∞-category and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then the *fiber* of f is a fiber sequence of the form:

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & Y \end{array}$$

Dually, a *cofiber* of f is a cofiber sequence of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

Remark A.18.6. As explained in previous remark, in the context of nerve of ordinary categories, fiber (cofiber) of f is kernel (cokernel) of f.

With these notions, we define the notion of stable ∞-categories.

Definition A.18.7. [Lur17, Definition 1.1.1.9] An ∞-category is *stable* if the following conditions hold:

1. $\mathcal{C}$ is pointed.

2. Every morphism in $\mathcal{C}$ admits fiber and cofiber.

3. A triangle in $\mathcal{C}$ is a fiber sequence iff it is a cofiber sequence.

Remark A.18.8. If $\mathcal{C} = N(C)$, then Remark A.18.2, Remark A.18.6 and Remark A.18.4 implies that the $\mathcal{C}$ is stable if and only if C has a zero object, a morphism in C admits a kernel and cokernel and $\mathrm{coim}(f) \cong \mathrm{im}(f)$ for any morphism $f : X \to Y$ in C.

Remark A.18.9. Some remarks on stable ∞-categories:

1. Any morphism in a pointed ∞-category admitting fibers and cofibers and it is determined uniquely upto equivalence. In particular, we have two maps:

$$\mathrm{fib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C} \text{ and } \mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$$

   which on the level of objects sends morphisms to its fiber and cofiber respectively.

2. The main examples of stable ∞-categories are derived ∞-categories ([Lur17, Definition 1.3.2.7]) and the ∞-category of spectra ([Lur17, Definition 1.4.3.1]). The homotopy category of the derived ∞-category is the classical derived category.

As triangulated categories have the notion of shifts, we can define the notion of shift functors in pointed ∞-categories.

Definition A.18.10. [Lur17, Remark 1.1.2.6] Let $\mathcal{C}$ be a pointed ∞-category. We define the *suspension(loop) functor* $\Sigma : \mathcal{C} \to \mathcal{C}$ on the level of objects as:

$$\Sigma(X) := \mathrm{cofib}(X \to 0), \quad \Omega(X) := \mathrm{fib}(0 \to X).$$

Notation A.18.11. [Lur17, Notation 1.1.2.7] In a pointed ∞-category, we denote for an object $X \in \mathcal{C}$,

$$X[n] := \begin{cases} \Sigma^n X & n \geq 0 \\ \Omega^{-n} X & n < 0 \end{cases}$$

The following proposition suggests that stable ∞-categories are ∞-categorical generalizations of triangulated categories.

Proposition A.18.12. *[Lur17, Theorem 1.1.2.14] If $\mathcal{C}$ is a stable ∞-category, then Σ, Ω are equivalences. Morever, the homotopy category $h\mathcal{C}$ is a triangulated category.*

In context of triangulated categories, we have the notion of exact functors which are functors between categories mapping distinguished triangles to distinguished triangles. We have an analog notion in the setting of stable ∞-categories.

Definition A.18.13. [Lur17, Section 1.1.4] Let $\mathcal{C}, \mathcal{C}'$ be stable ∞-categories and let $F : \mathcal{C} \to \mathcal{C}'$ sending zero objects to zero objects. Then we say F is *exact* if F carries fiber sequences to fiber sequences.

Notation A.18.14. We denote $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ to be the full subcategory of Cat_∞ whose objects are stable ∞-categories and morphisms are exact functors.

Proposition A.18.15. *[Lur17, Theorem 1.1.4.4] The ∞-category $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ admits all small limits and filtered colimits.*

Now we restrict ourselves to the setting of presentable stable ∞-categories. At first, we have the following lemma:

Lemma A.18.16. *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism in Pr^L where $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories. Then F is an exact funtor.*

Proof. As F is a morphism in Pr^L, then F by definition preserves all colimits. Thus F preserves cofiber sequences, hence F is exact. □

Notation A.18.17. We denote $\mathrm{Pr}^L_{\mathrm{stb}} \subset \mathrm{Pr}^L$ the full subcategory of presentable stable ∞-categories.

Remark A.18.18. By Proposition A.18.15, we see that $\mathrm{Pr}^L_{\mathrm{stb}}$ admits all limits.

APPENDIX B

ALGEBRA OBJECTS IN HIGHER CATEGORY THEORY

In this chapter, we recall the notion of ∞-operads and notion of algebra and module objects associated to an ∞-operad defined in [Lur17]. We also recall the fact that the category of presentable stable ∞-categories admits a symmetric monoidal structure. This shall led us to define the category of module objects $\mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{stb}})$ which is needed for defining the enhanced operation map.

B.1 ∞-operads.

A symmetric monoidal category is a category $\mathcal{C}$ with an unit object 1 and a functor: $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with the unital, commutativity and associativity constraints. The main problem in this definition is that we have to keep track of isomorphisms all the time while dealing these objects. In the setting of ∞-categories, these conditions even get more complicated.

The notion of ∞-operads help us to encode these relations in a coherent manner. In the language of higher category theory, symmetric monoidal ∞-categories are specific examples of ∞-operads. We recall the definition and properties of ∞-operads. These notions are motivated from the classical notion of colored operads (see [Lur17, Definition 2.1.1.1])
At first, we recall the definition of category of pointed finite sets Fin_*.

Definition B.1.1. [Lur17, Notation 2.0.0.1] The category Fin_* is defined as follows:

1. Objects: $\langle n \rangle = \{*, 1, 2, \dots, n-1, n\}$ where $\langle 0 \rangle = \{*\}$. Denote $\langle n \rangle^0 = \langle n \rangle - \{*\}$.
2. Morphisms: $\alpha : \langle n \rangle \to \langle m \rangle$ is a map of finite sets $\langle n \rangle^0 \to \langle m \rangle^0$ and $\alpha(*) = *$.

Remark B.1.2. 1. For every $1 \leq i \leq n$, let $\rho^i : \langle n \rangle \to \langle 1 \rangle$ be the map given the formula:

$$\rho^i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

2. Note that a morphism $\alpha : \langle n \rangle \to \langle m \rangle$ in Fin^* may take elements of $\langle n \rangle$ other than $*$ to $*$. Also there is no ordering condition in the morphism.

Definition B.1.3. [Lur17, Definition 2.1.1.8] We say a morphism $f : \langle m \rangle \to \langle n \rangle$ in Fin_* is *inert* if for $i \in \langle n \rangle^0$, the inverse image $f^{-1}\{i\}$ has exactly one element.

Definition B.1.4. [Lur17, Definition 2.1.10] An ∞-operad is a functor $p : \mathcal{O}^{\otimes} \to N(\mathrm{Fin}_*)$ between ∞-categories which satisfy the following conditions:

1. For every inert morphism $f : \langle m \rangle \to \langle n \rangle$ and every object $C \in \mathcal{O}^{\otimes}_{\langle m \rangle}$, there exists a p-coCartesian morphism $\bar{f} : C \to C'$ lifting f where $C' \in \mathcal{O}^{\times}_{\langle n \rangle}$. In particular f induces a functor $f_! : \mathcal{O}^{\otimes}_{\langle m \rangle} \to \mathcal{O}^{\otimes}_{\langle n \rangle}$.

2. Let $f : \langle m \rangle \to \langle n \rangle$ be a morphism in Fin_* and $C \in \mathcal{O}^{\otimes}_{\langle m \rangle}$ and $C' \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ be objects. Let $\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(C, C')$ be the union of the connected components of the simplicial set $\mathrm{Map}_{\mathcal{O}^{\otimes}}(C, C')$ which lie over f. Choose p-coCartesian lifts $C' \to C'_i$ over morphisms ρ^i(which are inert). Then:

$$\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(C, C') \to \prod_{i=1}^{n} \mathrm{Map}^{\rho^i \circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)$$

is a homotopy equivalence.

3. For every finite collection of objects $C_1 \cdots, C_n \in \mathcal{O}^{\otimes}_{\langle 1 \rangle}$, there exists an object $C \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ and a collection of p-coCartesian morphisms $C \to C_i$ covering $\rho^i : \langle n \rangle \to \langle 1 \rangle$.

By abuse of notation, $\mathcal{O}^{\otimes}$ shall be called an ∞-operad with the functor p and the properties implicitly assumed.

Remark B.1.5. [Lur17, Remark 2.1.1.12] We shall denote $\mathcal{O}^{\otimes}_{\langle 1 \rangle} = \mathcal{O}$ as the *underlying ∞-category of $\mathcal{O}^{\times}$.*

Example B.1.6. Below we state some examples of ∞-operads:

1. We shall denote the simplicial set $N(\mathrm{Fin}_*)$ by $\mathrm{Comm}^{\otimes}$. We call this the *commutative ∞-operad.*

2. We shall denote the simplicial subset of $N(\mathrm{Fin}_*)$ spanned by inert morphisms by $\mathrm{Triv}^{\otimes}$. We call this as the *trivial ∞-operad.*

We now proceed defining map of ∞-operads.

Definition B.1.7. [Lur17, Definition 2.1.2.7] Let $\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}$ be ∞-operads. A map of ∞-operads is a map of simplicial sets $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ satisfying the following conditions:

1. The diagram

$$\begin{array}{ccc} \mathcal{O}^{\otimes} & \xrightarrow{f} & \mathcal{O}'^{\otimes} \\ & \searrow \quad \swarrow & \\ & N(\mathrm{Fin}_*) & \end{array}$$

commutes.

2. f sends inert morphisms in $\mathcal{O}^{\otimes}$ to inert morphisms in $\mathcal{O}'^{\otimes}$. (Inert morphisms in $\mathcal{O}^{\otimes}$ are those morphisms which when mapped to $N(\mathrm{Fin}_*)$ are inert.)

Definition B.1.8. [Lur17, Definition 2.1.2.10] A map of operads $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ is said to be a *fibration* of ∞-operads if f is a categorical fibration.

Definition B.1.9. [Lur17, Definition 2.1.2.13] Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a categorical fibration of ∞-categories and $\mathcal{O}^{\otimes}$ be an ∞-operad. Denote q to be the composition $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to N(\mathrm{Fin}_*)$. If q exhibits $\mathcal{C}$ as an ∞-operad, we say p is a *coCartesian fibration of ∞-operads*.
We also say p exhibits $\mathcal{C}$ as an $\mathcal{O}$-monoidal category.

Definition B.1.10. [Lur17, Example 2.1.2.18] A *symmetric monoidal ∞-category* is an ∞-category $\mathcal{C}^{\otimes}$ equipped with a coCartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$. We will refer to the fiber $\mathcal{C}^{\otimes}_{\langle 1 \rangle}$ as the *underlying category of* $\mathcal{C}^{\otimes}$ which shall often be denoted by $\mathcal{C}$.

Remark B.1.11. There is another way to formulate the notion of symmetric monoidal ∞-category. A symmetric monoidal ∞-category is a coCartesian fibration of simplicial sets $p : \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ with the following property:
For each $n \geq 0$, the maps $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}$ induce functors $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C} := \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ which determine an equivalence $\mathcal{C}^{\otimes}_{\langle n \rangle} \to (\mathcal{C})^n$.

In particular, the map $\rho^1 : \langle 2 \rangle \to \langle 1 \rangle$ produces a map $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which satisfies associativity and other conditions upto homotopy.

Example B.1.12. Let $\mathcal{C} = \mathrm{Vect}_k$ to be the category of k-vector spaces. Then one can define a category $\widetilde{\mathrm{Vect}_k}$ as follows:

1. Objects are finite sequences of vector spaces $(V_!, V_2, \cdots, V_n)$.

2. A morphism between $(V_1, V_2, \cdots, V_n) \to (W_1, W_2, \ldots, W_m)$ consists of a map $\alpha : \langle n \rangle \to \langle m \rangle$ and a collection of morphisms

$$\{\phi_j \in \mathrm{Map}(\otimes_{i \in I, i \in \alpha^{-1}(j)} V_i, W_j)\}_{1 \leq j \leq m}.$$

Then the forgetful map $N(\widetilde{\mathrm{Vect}_k}) \to N(\mathrm{Fin}_*)$ makes $N(\widetilde{\mathrm{Vect}_k})$ into a symmetric monoidal ∞-category. Similarly, it is possible to visualize any symmetric monoidal category as a symmetric monoidal ∞-category.

B.2 Algebra objects.

We now move to defining algebra objects associated to an ∞-operad. Recall that vector spaces V are infact free modules over the field k. V can be an algebra too i.e. V has a ring structure. More formally, V is an algebra if it has an identity element i.e. $1 \to V$ and a map $V \times V \to V$. An algebra object associated to an ∞-operad is a generalization of the statement above.

Definition B.2.1. [Lur17, Definition 2.1.3.1] Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. Let $\alpha : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of ∞-operads. Define $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ to be the full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C})$ spanned by maps of ∞-operads. An object of $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ is said to be an *algebra object associated to the ∞-operad.*
If $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes}$, then we denote $\mathrm{Alg}_{\mathcal{O},/\mathcal{O}}(\mathcal{C})$ as $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.
If $\mathcal{O}'^{\otimes} = \mathcal{O}^{\times} = \mathrm{N}(\mathrm{Fin}_*)$, we denote $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ as $\mathrm{CAlg}(\mathcal{C})$. We refer $\mathrm{CAlg}(\mathcal{C})$ as the ∞-*category of commutative algebra objects of* $\mathcal{C}$.

Remark B.2.2. In the example of the ∞-operad $\mathrm{N}(\widetilde{\mathrm{Vect}_k}) \to \mathrm{N}(\mathrm{Fin}_*)$, an algebra object is equivalent to giving a vector space V which is a k-algebra.

We briefly recall of how to associate symmteric monoidal ∞-categories from ∞-categories admitting finite coproducts. At first, given an ∞-category $\mathcal{C}$, we associate an ∞-operad $\mathcal{C}^{\coprod}$. This turns out to be symmetric monoidal in case it admits finite coproducts.

Notation B.2.3. [Lur17, Construction, 2.4.3.1] We define a category Γ^* as follows:

1. Objects: Pairs $(\langle n \rangle, i)$ where $i \in \langle n \rangle^0$.
2. A morphism in Γ^* from $(\langle m \rangle, i)$ to $(\langle n \rangle, j)$ is a map $\alpha : \langle m \rangle \to \langle n \rangle$ where $\alpha(i) = j$.

Given a simplicial set $\mathcal{C}$, we define a new simplicial set $\mathcal{C}^{\amalg}$ equipped with a map $\mathcal{C}^{\coprod} \to \mathrm{N}(\mathrm{Fin}_*)$ with the following universal property. For any simplicial set K with a map $K \to \mathrm{N}(\mathrm{Fin}_*)$, we have the following bijection of Hom-sets:

$$\mathrm{Hom}_{\mathrm{N}(\mathrm{Fin}_*)}(K, \mathcal{C}^{\amalg}) = \mathrm{Hom}_{\mathrm{Sets}_\Delta}(K \times_{\mathrm{N}(\mathrm{Fin}_*)} \mathrm{N}(\Gamma^*), \mathcal{C}).$$

Theorem B.2.4. *[Lur17, Proposition 2.4.3.3] Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}^{\amalg}$ is an ∞-operad.*

Corollary B.2.5. *Let $\mathcal{C}$ be an ∞-category, then $\mathcal{C}^{\amalg}$ is a symmetric monoidal ∞-category iff $\mathcal{C}$ admits finite coproducts.*

Remark B.2.6. [Lur17, Remark 2.4.3.4] There is a dual theory for ∞-categories admitting finite products and Cartesian ∞-operads. In that case, we define a new simplicial set $\mathcal{C}^{\times}$ which is a symmetric monoidal ∞-category when $\mathcal{C}$ admits finite products. The two constructions $\mathcal{C}^{\times}$ and $\mathcal{C}^{\otimes}$ are canonically equivalent once the ambient ∞-category has zero object and admits finite products and coproducts.

The following theorem illustrates an important property of the functor $(-)^{\amalg}$

Theorem B.2.7. *[Lur17, Corollary 2.4.3.10, Theorem 2.4.3.18] Let $\mathcal{C}$ be an ∞-category admitting finite coproducts which is realized as the underlying ∞-category of $\mathcal{C}^{\amalg}$. Then*

1. *There is an equivalence of ∞-categories $\mathrm{CAlg}(\mathcal{C}) \cong \mathcal{C}$.*
2. *For every ∞-operad $\mathcal{D}$, a functor $\mathcal{C} \to \mathrm{CAlg}(\mathcal{D})$ is induced from a functor of ∞-operads $\mathcal{C}^{\amalg} \to \mathcal{D}^{\otimes}$*

Theorem B.2.8. *[Lur17, Proposition 3.2.2.1] Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category whose underlying ∞-category $\mathcal{C}$ admits limits. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. Then the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ admits limits.*

B.3 Symmetric monoidal structure on $\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}$

In this section, we recall results relating to the notion of symmetric monoidal structure on the ∞-category of $\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}$. This comes from the notion of tensor product for presentable ∞-categories. The definition of tensor product is technical. The explicit construction is discussed in [Lur17, Section 4.8].

Notation B.3.1. [Lur17, Proposition 4.8.1.17]. Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories. Then we denote:

$$\mathcal{C} \otimes \mathcal{D} := \mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$$

Remark B.3.2. The tensor product is defined in [Lur17, Section 4.8] via defining the notion of tensor products on the category of presheaves and ind-objects.

The bifunctor

$$\otimes : \mathrm{Pr}^{\mathsf{L}} \times \mathrm{Pr}^{\mathsf{L}} \to \mathrm{Pr}^{\mathsf{L}}$$

preserves colimits separately in each variable. Also, we get that Pr^{L} is a closed monoidal category with internal hom given by $\mathrm{LFun}(\mathcal{C}, \mathcal{D})$ for $\mathcal{C}$ and $\mathcal{D}$ presentable ∞-categories.

The following corollary explictly describes what are the commutative algebra objects of $\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}$.

Corollary B.3.3. *[Lur17, Corollary 4.8.2.19] There exists a symmetric monoidal structure on $\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}$. The commutative algebra objects of $\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}}$ are symmetric monoidal ∞-categories which are presentable, stable and the tensor product preserves colimits in each variable. The ∞-category* Sp *is an initial object in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathsf{L}}_{\mathsf{stb}})$.

B.4 Module objects.

This section is devoted to defining module objects associated to an ∞-operad. More precisely, given a fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, the previous section defines the ∞-category of algebra objects $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$. Given any object $\mathsf{A} \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, there is a notion of module objects over A which form an ∞-category $\mathrm{Mod}^{\mathcal{O}}_{\mathsf{A}}(\mathcal{C})$. More precisely, there exists a map of ∞-operads

$$\mathrm{Mod}^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$$

In this section, we define the underlying category of module objects of a symmetric monoidal ∞-category via defining algebra objects with respect to an ∞-operad **Pf**.

Definition B.4.1. [Rob14, Section 9.4.1.2] We define the colored operad **Pf** as follows:

1. The objects are two colors **a** and **m**.

2.

$$\mathrm{Mul}_{\mathbf{Pf}}(\{X_i\}_{i\in I}, Y) = \begin{cases} \{*\} \ \text{ if } X_i = \mathbf{a} \ \ \forall i \in I \ , \ Y = \mathbf{a} \\ \{*\} \ \text{ if } \exists j \in I \text{ with } X_j = \mathbf{m} \text{ and } X_i = \mathbf{a} \ \forall i \in I - \{j\} \ , \ Y = \mathbf{m} \\ \phi \ \ \text{otherwise} \end{cases}$$

Notation B.4.2. [Rob14, Section 9.4.1.2] We denote $\mathrm{Pf}^{\otimes}$ to be the operadic nerve of **Pf**. In other words,

$$\mathrm{Pf}^{\otimes} := \mathrm{N}^{\otimes}(\mathbf{Pf}).$$

It is an ∞-operad because **Pf** as a simplicial colored operad is fibrant.

Definition B.4.3. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category with the underlying ∞-category to be $\mathcal{C}$. Then we define

$$\mathrm{Mod}(\mathcal{C}) := \mathrm{Mod}^{\mathrm{Comm}}(\mathcal{C}) := \mathrm{Alg}_{\mathrm{Pf}^{\otimes}}(\mathcal{C})$$

Remark B.4.4. There is a general definition of module objects $\mathrm{Mod}^{\mathcal{O}^{\otimes}}(\mathcal{C})^{\otimes}$ with respect to a fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$. This is defined in [Lur17, Chapter 3, Section 3.3]. For the sake of our purposes, we just need to know the underlying ∞-category of module objects when $\mathcal{O}^{\otimes} := \mathrm{Comm}$.

In the rest of the section, we explain how a morphism $A \to B$ of commutative algebra objects in $\mathrm{CAlg}(\mathcal{C})$ visualizes the pair (A, B) as an object in $\mathrm{Mod}(\mathcal{C})$. At first, we give an alternate description of the ∞-operad $\Delta[1]^{\amalg}$.

Lemma B.4.5. *[Lur17, 2.4.3.1] The objects and morphisms of the ∞-operad $\Delta[1]^{\amalg}$ can be described in the following way:*

1. *Objects are pairs $(\langle n\rangle, S)$ where S is a subset of $\langle n\rangle^{+}$.*

2. *Morphisms from $(\langle n\rangle, S) \to (\langle m\rangle, T)$ is a morphism $\alpha : \langle n\rangle \to \langle m\rangle$ such that $\alpha(S) \subset T \cup \{0\}$.*

Definition B.4.6. We define a map of ∞-operads $u_{\mathrm{pf}} : \mathrm{Pf}^{\otimes} \to \Delta[1]^{\amalg}$ as follows:
We identify $\mathrm{Pf}^{\otimes}$ as a subsimplicial set of $\Delta[1]^{\amalg}$ by identifying $\mathbf{a} = 0$ and $\mathbf{m} = 1$. For morphisms, we only consider the morphisms $\alpha : \langle n\rangle \to \langle m\rangle$ such that $\alpha^{-1}(T) \cap S \to T$ is a bijection.

Let $f : A \to B$ be a morphism of commutative algebra objects in a symmetric monoidal ∞-category. Then we can visualize f as an object in $\mathrm{Fun}(\Delta[1], \mathrm{CAlg}(\mathcal{C}))$. By the universal property of $\Delta[1]^{\amalg}$, f corresponds uniquely to a morphism $\Delta[1]^{\amalg} \to \mathcal{C}^{\otimes}$. Combining with u_{pf}, we get a map

$$\mathrm{Pf}^{\otimes} \to \mathcal{C}^{\otimes}$$

This yields us an object in $\mathrm{Mod}(\mathcal{C})$. On the level of objects, the morphism f is sent to the pair (A, B).

B.5 Inversion of objects in symmetric monoidal ∞-categories.

In this section, we briefly recall about inversion of objects in symmetric monoidal ∞-categories. More precisely, given any symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ and given X an object in $\mathcal{C}^{\otimes}$, we can associate a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}[\mathsf{X}^{-1}]$ and a symmetric monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}[\mathsf{X}^{-1}]$ with the fact that this functor sends X to an invertible object in $\mathcal{C}^{\otimes}[\mathsf{X}^{-1}]$ and it satisfies the universal property with respect to inverting X.

The construction of $\mathcal{C}^{\otimes}[\mathsf{X}^{-1}]$ is explained in detail in [Rob14, Chapter 4]. In the case of stable homotopy theory, the inversion can be defined using the notion of spectrum objects in [Rob14, Chapter 4, Section 4.2].

We begin by recalling the notion of invertible objects, and then defining $\mathcal{C}^{\otimes}[\mathsf{X}^{-1}]$ and end with stating the universal property.

Definition B.5.1. [Rob14, Section 4.1] Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category and $\mathsf{X} \in \mathcal{C}$ an object, then X is said to be *invertible* if there exists $\mathsf{X}^* \in \mathcal{C}$ such that $\mathsf{X} \otimes \mathsf{X}^* \cong \mathsf{X}^* \otimes \mathsf{X} \cong \mathsf{u}$ where u is the unit object of $\mathcal{C}$.

Turns out that if there is some condition on the object X, the category $\mathcal{C}^{\otimes}[\mathsf{X}^{-1}]$ can be defined quite conceretly. We need X to be "symmetric". This defined below.

Definition B.5.2. [Rob14, Definition 4.2.7] Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. An object $\mathsf{X} \in \mathcal{C}$ is said to be symmetric if there exists a 2-equivalence between the cyclic permutation of $\mathsf{X} \otimes \mathsf{X} \otimes \mathsf{X}$ and the identity permutation of $\mathsf{X} \otimes \mathsf{X} \otimes \mathsf{X}$, i.e. there exists a 2-simplex in $\mathcal{C}$

$$\begin{array}{ccc} \mathsf{X}\otimes\mathsf{X}\otimes\mathsf{X} & \xrightarrow{\sigma} & \mathsf{X}\otimes\mathsf{X}\otimes\mathsf{X} \\ \downarrow{\scriptstyle \mathrm{id}} & \nearrow_{\mathrm{id}} & \\ \mathsf{X}\otimes\mathsf{X}\otimes\mathsf{X} & & . \end{array} \tag{B.1}$$

With this notion, one can define the underlying category $\mathcal{C}[\mathsf{X}^{-1}]$ of the symmetric monoidal ∞-category $\mathcal{C}^{\otimes}[\mathsf{X}^{-1}]$.

Notation B.5.3. Let $\mathcal{C}$ be symmetric monoidal ∞-category. Then we define:

$$\mathcal{C}[\mathsf{X}^{-1}] := \operatorname{colim}_{\mathrm{Mod}_{\mathcal{C}^{\otimes}}(\mathrm{Cat}_\infty)}(\cdots \xrightarrow{\otimes \mathsf{X}} \mathcal{C} \xrightarrow{\otimes \mathsf{X}} \mathcal{C} \xrightarrow{\otimes \mathsf{X}} \cdots) \tag{B.2}$$

The following is the main proposition of this section.

Proposition B.5.4. *[Rob14, Corollay 4.2.13] There exists a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}[\mathsf{X}^{-1}]$ whose underlying ∞-category is $\mathcal{C}[\mathsf{X}^{-1}]$ defined in Notation B.5.3. The canonical map $\mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}[\mathsf{X}^{-1}]$ sends X to an invertible object and the ∞-category $\mathcal{C}^{\otimes}[\mathsf{X}^{-1}]$ is universal with respect to this property.*

Remark B.5.5. Some remarks on the above proposition are as follows:

1. In [Rob14, Chapter 4, Section 4.1], one defines the symmetric monoidal ∞-category $\mathcal{C}^{\otimes}[\mathsf{X}^{-1}]$ by using the notion of $\mathrm{Free}_{\mathcal{C}^{\otimes}}(-)$.

2. The proof of Proposition B.5.4 follows from a statement of stabilization on the level of ordinary symmetric monoidal categories. For a symmetric monoidal category C and $\mathsf{X} \in \mathsf{C}$ be an object with some conditions analogue to symmetric, one can define $\mathrm{Stab}_{\mathsf{X}}(\mathsf{C})$ as the colimit of tensoring via X in the similar fashion. It satisfies the similar properties. This is actually [Rob15, Theorem 4.2.5].

APPENDIX C

MOTIVIC STABLE HOMOTOPY THEORY OF SCHEMES

In this chapter, we recall the definition and six operations of motivic stable homotopy theory of schemes explained in [Rob14].

C.1 Unstable $\mathbb{A}^1$-Homotopy Theory of Schemes.

In this section, we define the unstable motivic homotopy theory of a Noetherian scheme S with finite Krull dimension.
The main idea is to do "homotopy theory with schemes". As the category of schemes does not admit all colimits, we enlarge the category and also we want to make sure that $\mathbb{A}^1$ which is the analogue of $[0,1]$ in the world of algebraic geometry is contractible. The category $\mathcal{H}(S)$ these conditions. We briefly recall the construction (see [Rob14, Section 5.1] for details) as follows:

1. Let S be a Noetherian scheme of finite Krull dimension. Let $\mathrm{Sm}_S^{\mathrm{ft}}$ be the category of smooth schemes of finite type over S. Consider the $(\infty,1)$-category $\mathsf{N}(\mathrm{Sm}_S^{\mathrm{ft}})$.

2. We consider the Nisnevich topology on $\mathrm{Sm}_S^{\mathrm{ft}}$. The pairs $(V \to X, U \to X)$ form a basis of Nisnevich topology where the above morphisms form the Nisnevich distinguished square. This topology forms an $(\infty,1)$-site in $\mathsf{N}(\mathrm{Sm}_S^{\mathrm{ft}})$.

3. We consider the very big $(\infty,1)$-category of presheaves
$$\mathcal{P}^{\mathrm{big}}(\mathsf{N}(\mathrm{Sm}^{\mathrm{ft}}(S))) := \mathrm{Fun}(\mathsf{N}(\mathrm{Sm}^{\mathrm{ft}}(S)), \hat{\mathcal{S}}).$$
By [Lur09, Section 5.1], it is the free completion of $\mathsf{N}(\mathrm{Sm}_S^{\mathrm{ft}})$. By the Yoneda embedding $j : \mathsf{N}(\mathrm{Sm}_S^{\mathrm{ft}}) \to \mathcal{P}^{\mathrm{big}}(\mathsf{N}(\mathrm{Sm}_S^{\mathrm{ft}}))$, we identify a scheme X with $j(X)$.

4. We restrict to those presheaves which are sheaves with respect to the Nisnevich topology as described in the second point. We denote $\mathrm{Sh}_{\mathrm{Nis}}^{\mathrm{big}}(\mathrm{Sm}_S^{\mathrm{ft}}) \subset \mathcal{P}^{\mathrm{big}}(\mathsf{N}(\mathrm{Sm}_S^{\mathrm{ft}}))$ be the very

big $(\infty,1)$-category of sheaves with respect to the Nisnevich topology. An example of sheaves are the representable sheaves $j(X)$ for any scheme $X \in \mathrm{Sm}_S^{\mathrm{ft}}$.

5. The last step of this construction is to invert the affine line $\mathbb{A}^1$. For this we want to restrict ourselves to sheaves which satisfy $\mathbb{A}^1$-invariance i.e. sheaves F which for every scheme X, satisfy $F(X) \to F(X \times \mathbb{A}^1)$ is an equivalence. This is achieved by localization process. We localize with respect to class of projection maps $\{(X \times \mathbb{A}^1) \to X\}$. Let $\mathcal{H}(S)$ be the localization of $\mathrm{Sh}_{\mathrm{Nis}}^{\mathrm{big}}(\mathrm{Sm}_S^{\mathrm{ft}})$ with respect to the class of these projection maps.

Remark C.1.1. The above construction is functorial in $\mathrm{Sch}_{\mathrm{fd}}$ and thus we have a functor

$$\mathcal{H} : \mathrm{Sch}_{\mathrm{fd}}^{\mathrm{op}} \to \widehat{\mathrm{Cat}_\infty}$$

where $\mathrm{Sch}_{\mathrm{fd}}$ is the nerve of the category of Noetherian schemes with finite Krull dimension. It is an ∞-sheaf when $\mathrm{Sch}_{\mathrm{fd}}$ is equipped with Nisnevich topology ([Hoy17, Proposition 4.8]).

C.2 The $(\infty,1)$-category $\mathcal{H}(S)_*^{\wedge}$.

Notation C.2.1. Denote $\mathcal{H}(S)_*$ to be the $(\infty,1)$-category of pointed motivic objects. This makes sense as $\mathcal{H}(S)$ is a presentable $(\infty,1)$-category and thus admits a final object.

Thus, there exists a pair of adjoint functors:

$$()_+ : \mathcal{H}(S) \leftrightarrow \mathcal{H}(S)_* : \mathrm{Frgt}$$

where $()_+$ is defined as $()_+ : X \to X_+ := X \coprod *$ and the other is the forgetful functor.

There is a technical lemma which helps us to inherit the cartesian product in $\mathcal{H}(S)$ to a symmetric monoidal structure on $\mathcal{H}(S)_*$. The following proposition gives us the way to achieve the following.

Proposition C.2.2. *[Rob14, Corollary 5.2.1] The functor*

$$()_* : \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{pt}}$$

which is a left adjoint to the inclusion functor $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{pt}} \hookrightarrow \mathrm{Pr}^{\mathrm{L}}$ induces a functor

$$()_*^{\wedge} : \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{pt}})$$

which is a left adjoint to the inclusion functor. The functor on the underlying ∞-categories is the functor $()_*$.

Notation C.2.3. [Rob14, Remark 5.2] Let $\mathcal{C}$ be an ∞-category which admits finite products. Then we denote the symmetric monoidal ∞-category

$$\mathcal{C}_*^{\wedge} := (\mathcal{C}^{\times})_*^{\wedge}$$

where $\mathcal{C}^{\times}$ is the symmetric monoidal ∞-category associated to the ∞-category $\mathcal{C}^{\otimes}$.

Definition C.2.4. We denote the pointed motivic homotopy category

$$\mathcal{H}(S)^{\wedge}_{*} := (\mathcal{H}(S)^{\times})^{\wedge}_{*} \tag{C.1}$$

whose underlying ∞-category is $\mathcal{H}(S)_*$ ([Lur17, Proposition 2.4.1.5 (4)]).

Remark C.2.5. The definition of $\mathcal{H}^{\wedge}_{*}$ is functorial in $\mathrm{Sch}_{\mathrm{fd}}$. Thus

$$\mathcal{H}^{\wedge}_{*} : \mathrm{Sch}^{\mathrm{op}}_{\mathrm{fd}} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}_{\infty}})$$

C.3 The stable motivic homotopy theory.

The stable motivic homotopy theory is defined by inverting the $(\mathbb{P}^1, \infty)$ spectrum in the ∞-category $\mathcal{H}(S)^{\wedge}_{*}$. The definition is as follows:

Definition C.3.1. Let S be a base scheme. The *stable motivic* $\mathbb{A}^1\infty$*-category* over S is the underlying ∞-category of the stable presentable symmetric monoidal ∞-category $\mathcal{SH}^{\otimes}(S)$ defined by the formula

$$\mathcal{SH}^{\otimes}(S) := \mathcal{H}^{\wedge}_{*}(S)[(\mathbb{P}^1, \infty)^{-1}] \tag{C.2}$$

It is denoted by $\mathcal{SH}(S)$. Here the inversion of $(\mathbb{P}^1, \infty)$ is in the sense of [Rob14, Chapter 4] which we recalled in Proposition B.5.4.

Remark C.3.2. Some remarks on the definition are as follows:

1. The homotopy category $h\mathcal{SH}^{\otimes}(S)$ coincides with Morel Voevodsky's definition of stable motivic category. This is proved in [Rob14, Theorem 4.3.1].

2. The pointed space $(\mathbb{P}^1, \infty)$ is symmetric and hence inverting the object gives a stable ∞-category.

C.4 Functoriality and six operations of $\mathcal{SH}^{\otimes}(S)$

In this section, we state functoriality of $\mathcal{SH}^{\otimes}$ and state the six operations. The main reference is [Rob14, Chapter 9]. The results are already proven in the level of homotopy categories of $\mathcal{SH}^{\otimes}$ due to [Ayo07a],[Ayo07b]. We only state the formalism of six operations on the level of ∞-categories as described in [Rob14]. At first, we state proper and smooth base change. We also recall the localization property and briefly recall the construction of the transformation α_f and purity transformation ρ_f. We end the section by stating the results in a single theorem.

C.4.1 Smooth and proper base change.

Proposition C.4.1. *[Rob14, Example 9.4.6] For g a smooth morphism of schemes, we have a left adjoint of g^* denoted by $g_\#$. The smooth(proper) base change is the following statement: If*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{g'} & & \downarrow{g} \\ X & \xrightarrow{f} & Y \end{array} \tag{C.3}$$

is a cartesian square of base schemes with g *being smooth (proper). We have the following equivalences:*

$$\mathrm{Ex}(\Delta^*_{\#}) : f'^{*}g'_{\#} \cong g_{\#}f^{*} \quad (\mathrm{Ex}(\Delta^*_{*}) : f'^{*}g'_{*} \cong g_{*}f^{*}) \tag{C.4}$$

Remark C.4.2. We also have the exchange transformation $\mathrm{Ex}(\Delta^*_*) : f'_* g^* \cong g^* f_*$ which is an isomorphism if g and g' are smooth.

An equivalent way of formulating the projection formula is the following: If $f : S' \to S$ is a separated morphism of finite type, the map

$$f_! : \mathcal{SH}(S) \to \mathcal{SH}(S') \tag{C.5}$$

is a map of $\mathcal{SH}^{\otimes}(S)$- modules (Eq. (4.4)). The proof of this statement follows from the projection formula for smooth and proper maps:

Proposition C.4.3. *[Rob14, Example 9.4.2] (Projection Formula)Let* $f : Y \to X$ *a smooth(proper) map. For* $E, E' \in \mathcal{SH}(X)$ *and* $B \in \mathcal{SH}(Y)$*, we have the following equivalences:*

1. $$f_{\#}(B \otimes f^*(E)) \cong (f_{\#}B \otimes E) \tag{C.6}$$

   when f *is smooth.*

2. $$f_{*}(B \otimes f^*(E)) \cong (f_{*}B \otimes E) \tag{C.7}$$

   when f *is proper.*

Remark C.4.4. Assuming smooth and proper base change, one can prove projection formulas and base change for $f_!$. The procedure of proving such will be explained by the enhanced operation map in Appendix D.

C.4.2 Localization and homotopy invariance.

In this subsection, we recall the localization property and homotopy invariance. We restate the statement of localization property and homotopy invariance.

Proposition C.4.5. *[Rob14, Theorem 9.4.25] Let* $i : Z \to X$ *be a closed immersion of base schemes. Let* $U := X - Z$ *be complement of* Z *and* $j : U \to X$ *be the open immersion.*

1. *The pushforward* i_* *is conservative.*

2. *For any object* $E \in \mathcal{SH}(X)$*, the square*

$$\begin{array}{ccc} j_{\#}j^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_*i^*(E) \end{array} \tag{C.8}$$

   is a fiber/cofiber sequence. In other words, we say $\mathcal{SH}$ *satisfies localization property.*

Remark C.4.6. 1. This is a reformulation of classical localization property stated in [MV99].

2. A consequence of the localization property is the fact that i_* is fully faithful.

3. Being a fiber/cofiber sequence implies that for any morphism $u : E \to F$ in $\mathcal{SH}(S)$, then u is an equivalence iff $j^*(u), i^*(u)$ are equivalences.

We now restate the homotopy invariance property.

Theorem C.4.7. *[Rob14, Example 9.4.26] Let $\pi : \mathbb{A}^1_X \to X$ be the usual projection map, then π^* is fully faithful. We refer to this as $\mathcal{SH}$ satisfies homotopy invariance.*

Remark C.4.8. Some remarks on the homotopy invariance:

1. As π is smooth, via smooth projection formula verifying homotopy invariance is equivalent to verify $\pi_\#(\pi^*(1_X)) \to 1_X$ is an equivalence.

2. The homotopy invariance follows from the definition of $\mathbb{A}^1$ invariance imposed in definitions of $\mathcal{H}$ and $\mathcal{SH}$.

C.4.3 Construction of α_f and purity transformation ρ_f.

We recall the construction of the map α_f and the purity transformation ρ_f. This shall be important in constructing the six operations on the level of stacks.

1. **The morphism α_f:** Let $f : S' \to S$ be a morphism of schemes. Factorize $f = pj$ where $j : S' \to S''$ is open immersion and $p : S'' \to S$ is proper. At first, we have the commutative square:

$$\begin{array}{ccc} S' & \xrightarrow{\mathrm{id}} & S' \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle j} \\ S' & \xrightarrow{j} & S'' \end{array} \tag{C.9}$$

The smooth exchange transformation $\mathrm{Ex}(\Delta_{\#*})$ gives us a morphism

$$\mathrm{id}_* \, j_\# \to \mathrm{id}_\# \, j_*.$$

Thus we construct the natural transformation α_f as follows:

$$\alpha_f : f_! = p_* j_\# \to p_* j_* = f_*. \tag{C.10}$$

It is obvious via the definition of $f_!$ that α_f is an equivalence when f is proper.

2. **The purity transformation ρ_f:** Let $f : X \to S$ be a smooth morphism. We consider the cartesian square of schemes

$$\begin{array}{cccc} X & & & \\ & \searrow^{\delta} & & \\ & X \times_S X & \xrightarrow{p} & X \\ & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\ & X & \xrightarrow{f} & S. \end{array} \tag{C.11}$$

Then we define

$$(\mathrm{Tw}_f)^{-1} := \Sigma_f := p_{\#}\delta_*. \tag{C.12}$$

We construct the map ρ_f in two steps:

(a) If f is proper, then using $\mathrm{Ex}(\Delta_{\#*})$, we have

$$\alpha_f : f_{\#} \to f_{\#}q_*\delta_* \xrightarrow{\mathrm{Ex}(\Delta_{\#*})} f_*p_{\#}\delta_* = f_* \circ \Sigma_f. \tag{C.13}$$

(b) If f is separated morphism of finite type, there exists a similar exchange transformation $\mathrm{Ex}(\Delta_{*!})$, thus using the same argument above,we get the map:

$$\rho_f : f_{\#} \to f_!\Sigma_f.$$

Remark C.4.9. The above sketch of construction of α_f and ρ_f is explained in [CD19, Chapter 2].

Proposition C.4.10. *[Rob14, Theorem 9.4.38] The functor $\mathcal{SH}$ satisfies purity property for smooth and separated morphisms of finite type i.e.. the Thom transformation Σ_f and the natrual transformation ρ_f are equivalences.*

We also recall another description of the natural transformation Σ_f.

Theorem C.4.11. *[Rob14, Theorem 9.4.34] Let*

$$\begin{array}{ccc} Z & \overset{g}{\hookrightarrow} & X \\ & & \downarrow p \\ & & S \end{array} \tag{C.14}$$

where i is a closed immersion of smooth S-schemes. Then the deformation to the normal cone produces an equivalence

$$\frac{X}{X/Z} := p_{\#}(i_*(1_Z)) \cong (g \circ q)_{\#}(e_*(1_Z)) =: \mathrm{Th}_S(N_ZX)$$

in $\mathcal{SH}^{\otimes}(S)$ where $g : N_ZX \to X$ is the normal bundle and $e : Z \to N_ZX$ is the zero section.

The proof of the proposition is a consequence of the following proposition on the level of $\mathcal{H}(S)$.

Proposition C.4.12. *Consider the commutative diagram of schemes in Eq. (C.14). Then the canonical maps of smooth pairs:*

$$(X, Z) \xleftarrow{p_0} (D_ZX, \mathbb{A}^1_Z) \xrightarrow{p_1} (N_ZX, Z)$$

induces an equivalence:

$$\left(\frac{X}{X/Z}\right) \cong \left(\frac{N_ZX}{N_ZX/Z}\right) \tag{C.15}$$

in $\mathcal{H}(S)$.

The proof of proposition Theorem C.4.11 follows from Proposition C.4.12 with localization and homotopy property.

The proposition Theorem C.4.11 gives us the explicit description of Σ_f as follows:

$$\Sigma_f = p_{\#}\delta_* \cong p_{\#}\delta_*(1_X) \otimes_X (-) \cong \mathrm{Th}_X(N_f) \otimes_X (-) \tag{C.16}$$

where N_f is the tangent bundle of f.

C.4.4 Statement of six operations for $\mathcal{SH}^{\otimes}$.

Theorem C.4.13. *[Rob14, Theorem 9.4.8] There exists a functor*

$$\mathcal{SH}^{\otimes} : \mathrm{Sch}_{fd}^{op} \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{stb}) \tag{C.17}$$

and for a morphism $f : Y \to X$ we have the following functors:

1. $f^* : \mathcal{SH}^{\otimes}(X) \to \mathcal{SH}^{\otimes}(Y)$.
2. $f_* : \mathcal{SH}(Y) \to \mathcal{SH}(X)$.
3. $f_! : \mathcal{SH}(X) \to \mathcal{SH}(Y)$ *when f is separated and of finite type.*
4. $f^! : \mathcal{SH}(Y) \to \mathcal{SH}(X)$ *when f is separated of finite type.*
5. $- \otimes - : \mathcal{SH}(X) \otimes \mathcal{SH}(X) \to \mathcal{SH}(X)$.
6. $\mathrm{Hom}_{\mathcal{SH}(X)}(-,-) : \mathcal{SH}(X) \times \mathcal{SH}(X) \to \mathcal{SH}(X)$.

Morever, the functor $\mathcal{SH}^{\otimes}$ along with the functors $(f^*, f_*, f_!, f^!, - \otimes -, \mathrm{Hom}(-,-))$ satisfy the following properties:

1. *(Monoidality) f^* is monoidal,i.e. there exists an equivalence*

$$f^*(E \otimes E') \cong f^*(E) \otimes f^*(E') \tag{C.18}$$

   for $E, E' \in \mathcal{SH}^{\otimes}(X)$

2. *(Projection Formula) For $E, E' \in \mathcal{SH}(X)$ and $B \in \mathcal{SH}(Y)$, we have the following equivalences:*

   (a)

$$f_!(B \otimes f^*(E)) \cong (f_! B \otimes E) \tag{C.19}$$

   (b)

$$f^! \mathrm{Hom}_{\mathcal{SH}(X)}(E, E') \cong \mathrm{Hom}_{\mathcal{SH}(Y)}(f^* E, f^! E') \tag{C.20}$$

3. *(Base Change) If*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \tag{C.21}$$

   is a cartesian square of base schemes with g being separated of finite type, we have the following equivalences:

(a)
$$\mathrm{Ex}(\Delta^*_!) : \mathsf{f}'^*\mathsf{g}'_! \cong \mathsf{g}_!\mathsf{f}^* \tag{C.22}$$

(b)
$$\mathrm{Ex}(\Delta^!_*) : \mathsf{f}'_*\mathsf{g}'^! \cong \mathsf{g}^!\mathsf{f}_* \tag{C.23}$$

4. *(Proper pushforward) If* f *is a separated morphism of finite type, then there exists a natural transformation:*
$$\alpha_{\mathsf{f}} : \mathsf{f}_! \to \mathsf{f}_* \tag{C.24}$$
which is an equivalence if f *is proper.*

5. *(Purity) For* f *to be smooth morphism separated of finite type, there exists a self equivalence* Tw_{f} *and an equivalence*
$$\mathrm{Tw}_{\mathsf{f}} \circ \mathsf{f}^! \cong \mathsf{f}^*$$

6. $\mathcal{SH}^{\otimes}$ *satisfies Nisnevich descent.*

APPENDIX D

THE CONSTRUCTION OF THE ENHANCED OPERATION MAP

In this chapter, we review the construction of the enhanced operation map due to Liu and Zheng. In the first section, we recall the ∞-categorical statement of Deligne's glueing ([LZ12, Theorem 0.1]). We also give a brief review of the proof of the glueing and recall the notion of simplicial set of compactifications and cartesianizations. In the second section, we recall the theorem of partial adjoints ([LZ17, Proposition 1.4.4]). Using the theorems of glueing and partial adjoints, we give a brief review of the construction of the enhanced operation map in the last section as explained in [Rob14, Section 9.4.1.3].

D.1 Deligne's compactification in ∞-categories.

D.1.1 Motivation.

The extraordinary pushforward is one of the six functors involved in the six operations of étale cohomology of schemes and SH. For example, let $f : X \to Y$ be a separated morphism of finite type of quasi-compact and quasi-separated schemes and Λ be a torsion ring. We have the extraordinary pushforward map

$$f_! : \mathcal{D}(X, \Lambda) \to \mathcal{D}(Y, \Lambda)$$

which when restricted to open immersions is the map $f_\#$ and to proper morphisms the map f_*. The construction of $f_!$ involves the general theory of glueing two psuedofunctors developed by Deligne. In this chapter, we briefly review the theory of glueing in higher categories due to Liu and Zheng ([LZ12]). Before motivating the main theorem, let us briefly recall the classical argument of glueing due to Deligne.

For any morphism f as above, we consider the $(2, 1)$-category of compactifications $\mathrm{Sch}^{\mathsf{cmp}}$

whose objects are schemes and morphisms are triangles

$$\begin{array}{ccccc} & & \overline{Y} & & \\ & \overset{j}{\nearrow} & & \overset{p}{\searrow} & \\ X & & & & Y \end{array} \tag{D.1}$$

where j is open and p is proper. It is important to note that one can compose morphisms of such form due to Nagata's theorem of compactification. Then one can define a pseudo-functor $F_c : \mathrm{Sch}^{cmp} \to \mathrm{Cat}_1$ which sends a scheme X to $D(X, \Lambda)$ and a triangle of the form above to the composition $p_* \circ j_\#$ (here Cat_1 denotes the 2-category of categories). The theory of glueing in 2-categories tell us that the functor F_c can be extended to a functor $f_!$ from the category Sch' consisting of schemes where morphisms are separated and finite type. In other words, the diagram

$$\begin{array}{ccc} \mathrm{Sch}^{comp} & \xrightarrow{F_c} & \mathrm{Cat}_1 \\ \downarrow & \overset{f_!}{\nearrow} & \\ \mathrm{Sch}' & & \end{array} \tag{D.2}$$

commutes.

In the language of ∞-categories, we replace the category Sch^{comp} by a simplicial set $\delta_2^* N(\mathrm{Sch}')^{cart}_{P,O}$. The n-simplices of $\delta_2^* N(\mathrm{Sch}')_{P,O}$ are $n \times n$ grids of the form

$$\begin{array}{ccccccc} X_{00} & \longrightarrow & X_{01} & \longrightarrow & \cdots & & X_{0n} \\ \downarrow & & \downarrow & & & & \downarrow \\ X_{10} & \longrightarrow & X_{11} & \longrightarrow & \cdots & & X_{1n} \\ & & & & & & \\ \vdots & & \vdots & & \vdots & & \vdots \\ & & & & & & \\ X_{n1} & \longrightarrow & X_{n2} & & \cdots & & X_{nn} \end{array} \tag{D.3}$$

where vertical arrows are open, horizontal arrows are proper and each square is a pullback square. Also one has a natural morphism $p_{\mathrm{Sch}'} : \delta_2^* N(\mathrm{Sch}')_{P,O} \to N(\mathrm{Sch}')$ induced by composition along the diagonal. We can now state the result of glueing.

Theorem D.1.1. *[LZ12, Corollary 0.3] The natural map*

$$p_{\mathrm{Sch}'} : \delta_2^* N(\mathrm{Sch}')^{cart}_{P,O} \to N(\mathrm{Sch}')$$

is a categorical equivalence.

In particular, given any morphism $g : \delta_2^* N(\mathrm{Sch}')^{cart}_{P,O} \to \widehat{\mathrm{Cat}_\infty}$, there exists a morphism $\tilde{g} : N(\mathrm{Sch}') \to \widehat{\mathrm{Cat}_\infty}$ such that $p_{\mathrm{Sch}'} \circ \tilde{f}$ and f are homotopic (i.e. there exists a morphism $F : \Delta^1 \times \delta_2^* N(\mathrm{Sch}')_{P,O} \to \widehat{\mathrm{Cat}_\infty}$ such that $F|_{[0]} = p_{\mathrm{Sch}'} \circ \tilde{g}$ and $F|_{[1]} = g$). The above glueing statement shall play an important role in constructing the enhanced six operation map. The

enhanced six operation map shall encode $f_!$ and base change formula with respect to lower shriek functors.
The above theorem is a special case of Theorem D.1.3. This uses the notion of multi-marked and multi-tiled simplicial sets which we recall in Chapter 4.

D.1.2 Statement of the theorem.

At first, we recall the notion of admissible edges.

Definition D.1.2. Let C be an ordinary category. Let $\mathcal{E}$ be a collection of morphisms in C. Then $\mathcal{E}$ is said to be *admissible* if

1. $\mathcal{E}$ contains every identity morphism in C.
2. $\mathcal{E}$ is stable under pullbacks.
3. For every pair of composable morphisms $p \in \mathcal{E}$ an q a morphism in C, then if $p \circ q \in E$ implies $q \in E$.

Theorem D.1.3. *Let C be a category admitting pullbacks. Let $\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_k$ where $k \geq 2$ be sets of morphisms each containing identity morphism and satisfy the following conditions:*

1. *$\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{E}_0$ and $\mathcal{E}_1, \mathcal{E}_2$ are admissible.*
2. *Every morphism $f \in \mathcal{E}_0$ can be factorised as $f = p \circ q$ where $p \in \mathcal{E}_1$ and $q \in \mathcal{E}_2$.*
3. *For every $k \geq 3$, $\mathcal{E}_i$ is stable under pullbacks by $\mathcal{E}_1$.*

Then the natural map

$$p_c : \delta_k^* N(C)_{\mathcal{E}_1, \cdots, \mathcal{E}_k} \to \delta_{k-1}^* N(C)_{\mathcal{E}_0, \mathcal{E}_3, \cdots, \mathcal{E}_k}$$

is a categorical equivalence.

An immediate corollary is the following which is the ∞-categorical statement of Deligne's glueing.

Corollary D.1.4. *Let $C =$ Sch' be the category of quasi-compact and quasi-separated schemes with morphisms separated of finite type. Let P be the collection of proper morphisms and O be the collection of open immersions. Then the map*

$$p_C : \delta_2^* N(C)_{P,O} \to N(C)$$

is a categorical equivalence.

Remark D.1.5.
1. The corollary is proved by applying Theorem D.1.3 on $k = 2$ and $\mathcal{E}_0 = \mathrm{Ar}(\mathrm{Sch'}), \mathcal{E}_1 = P, \mathcal{E}_2 = O$.
2. Unwinding the definition of categorical equivalence, the theorem is equivalent to prove the following statement: Let $\mathcal{D}$ be an ∞-category. Then the functor:

$$\mathrm{Fun}(\delta_{k-1}^* N(C)_{\mathcal{E}_0, \mathcal{E}_3, \cdots, \mathcal{E}_k}, \mathcal{D}) \to \mathrm{Fun}(\delta_k^* N(C)_{\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_k}, \mathcal{D})$$

is an equivalence on the level of homotopy categories.

3. For the sake of simplicity, we shall give an idea of the proof the statement for $k = 2$ and $\mathcal{E}_0 = \mathrm{Ar}(C)$.

4. If $\mathcal{D}$ is the Duskin nerve of Cat_1 (the category of small categories), we get Deligne's result on the level of homotopy categories.

Notation D.1.6. We denote the map p_C in Theorem D.1.3 as the composition

$$\delta_2^* N(C)^{cart}_{\mathcal{E}_1,\mathcal{E}_2} \xrightarrow{p_{cart}} \delta_2^* N(C)_{\mathcal{E}_1,\mathcal{E}_2} \xrightarrow{p_{comm}} N(C)$$

The idea to prove the specified version of Theorem D.1.3 is by proving that p_{cart} and p_{comm} are categorical equivalences.

We shall use the following lemma to prove that a morphism of simplicial sets is a categorical equivalence.

Lemma D.1.7. *[LZ12, Lemma 1.9] A map of simplicial sets $f : Y \to Z$ is a categorical equivalence iff for any ∞-category $\mathcal{D}$, the following conditions hold:*

1. *For every $l = 0, 1$ and every commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{v} & \mathrm{Fun}(\Delta^l, \mathcal{D}) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{w} & \mathrm{Fun}(\partial\Delta^l, \mathcal{D}) \end{array} \tag{D.4}$$

   where p is induced by $\partial\Delta^l \subset \Delta^l$, there exists a map $u : Z \to \mathrm{Fun}(\Delta^l, \mathcal{D})$ such that $p \circ u = w$ and $u \circ f, v$ are homotopic over $\mathrm{Fun}(\partial\Delta^l, \mathcal{D})$.

2. *For $l = 2$ and for every commutative diagram in Eq. (D.4), there exists a map $u : Z \to \mathrm{Fun}(\Delta^l, \mathcal{D})$ and $p \circ u = w$.*

Remark D.1.8. We give a brief idea on proving Corollary D.1.4. The key is to use Lemma D.1.7. We give a sketch on proving each p_{cart} and p_{comm} satisfy the condition of lemma for $l = 0$.

1. For $l = 0$, we need to show the existence of the following dotted arrow:

$$\begin{array}{ccc} \delta_2^* N(C)_{P,O} & \xrightarrow{v} & \mathcal{D} \\ \downarrow{\scriptstyle p_{comm}} & \overset{u}{\nearrow} & \\ N(C) & & \end{array} \tag{D.5}$$

   On the level of objects, u is already defined as it is same as v. Let us give a brief sketch on how one defines u on the level of morphisms. Let $f : Y \to X$ be a morphism of schemes. Factorize f as $Y \xrightarrow{f'} Y' \xrightarrow{g'} X$ where f' is open and g' is proper. The following diagram:

$$\begin{array}{ccccc} Y & \longrightarrow & Y & & \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f'} & & \\ Y' & \longrightarrow & Y' & \xrightarrow{g'} & X \\ & & \downarrow & & \downarrow \\ & & Y' & \xrightarrow{g'} & X \end{array} \tag{D.6}$$

gives a morphism $u_{12} : \Lambda^2_1 \to \delta^*_2 N(C)_{P,O}$. This induces a map $\nu \circ u_{12} : \Lambda^2_1 \to \mathcal{D}$. As $\mathcal{D}$ is an ∞-category, this extends to a map $\nu : \Delta^2 \to \mathcal{D}$. We define $u(f) := \nu|_{\Delta^1}$(restricted to edge opposite to 1). Thus modulo the choice of compactification, we can define u on the level of morphisms.

2. For p_{cart}, one can proceed in the following way. We need to prove the existence of dotted arrow

$$\begin{array}{ccc} \delta^*_2 N(C)^{cart}_{P,O} & \xrightarrow{\nu} & \mathcal{D}. \\ \big\downarrow p_{cart} & \overset{u}{\nearrow} & \\ \delta^*_2 N(C)_{P,O} & & \end{array} \tag{D.7}$$

As before, u on the level of 0-simplicies is already defined via ν. Let us sketch how one can define u on the level of 1-simplices. Let σ

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \big\downarrow g' & & \big\downarrow g \\ Y & \xrightarrow{f} & X \end{array} \tag{D.8}$$

be a 1 simplex of $\delta^*_2 N(C)_{P,O}$ where f', f are proper and g', g are open. Then, we have a diagram

$$\begin{array}{ccccc} Y' & & & & \\ & \searrow^{h} & & & \\ & & Y \times_X X' & \xrightarrow{f_1} & X' \\ & & \big\downarrow g_1 & & \big\downarrow g \\ & & Y & \xrightarrow{f} & X \end{array} \qquad Y' \xrightarrow{f'} X', \quad Y' \xrightarrow{g'} Y \tag{D.9}$$

where the bottom square is Cartesian. The morphism h is open and proper as f', f_1 are proper and g, g_1 are open. Thus we also have a cartesian square:

$$\begin{array}{ccc} Y' & \xrightarrow{id} & Y' \\ \big\downarrow id & & \big\downarrow h \\ Y' & \xrightarrow{h'} & Y \times_X X' \end{array} \tag{D.10}$$

Combining these two cartesian squares, gives us a morphism

$$h' : \Lambda^2_1 \to \delta^*_2 N(C)^{cart}_{P,O} \xrightarrow{p_{cart}} \mathcal{D}$$

As $\mathcal{D}$ is an ∞-category, this extends to $h'_1 : \Delta^2 \to \mathcal{D}$. We define $u(\sigma) = h'_1|_{\Delta^1}$ (edge opposite to 1).

D.1.3 Simplicial set of compactifications and cartesianizations.

In this subsection, we define two important simplicial sets: simplicial set of compactifications and cartesianizations. The simplicial set of compactifications is an important tool for showing that the p_{com} is a categorical equivalence. The simplicial set of cartesianizations is need for showing that the p_{cart} is a categorical equivalence. The definitions of both of these are motivated from the ideas of proving the theorem.
We shall define specific simplicial sets which shall play a key role in defining these objects (see [LZ12, Section 4] and [LZ12, Section 5] for more details).

Definition D.1.9. [LZ12, Notation 4.1] Let $\mathcal{K}\mathrm{pt}^n$ be the sub-bisimplicial set of the bisimplicial set $\Delta^{n,n}$ spanned by vertices (i,j) where $1 \leq i \leq j \leq n$.

Definition D.1.10. Let $\mathrm{Cpt}^n \subset [n] \times [n]$ be the category spanned by objects $(i,j), 1 \leq i \leq j \leq n$. Denote $\mathcal{C}\mathrm{pt}^n := \mathrm{N}(\mathrm{Cpt}^n)$.

Notation D.1.11. Denote $\square^n := \delta_2^* \mathcal{K}\mathrm{pt}^n$. Also for a partially ordered set P with ordering $\leq$ and two elements $x, y \in P$, we denote:

1. $P_{x/}$ to be the undercategory of x.
2. $P_{/x}$ to be the overcategory of x.
3. $P_{x//y}$ to be the category spanned by objects $z \in P$ where $x \leq z \leq y$. It is empty if $x > y$.

Remark D.1.12. Some remarks on the definitions above.

1. Diagram of $\mathcal{C}\mathrm{pt}^1$ is as follows:

$$\begin{array}{ccc} a_{00} & \longrightarrow & a_{01} \\ & & \downarrow \\ & & a_{11} \end{array}$$

   Thus $\mathcal{C}\mathrm{pt}^1 \cong \Delta^2$. Here a_{ij} is the vertex (i,j) in $[n] \times [n]$.

2. We have a natural inclusion $\square^n \subset \mathcal{C}\mathrm{pt}^n$.
3. The Hasse diagram of $\square^1$ is as follows:

$$\square^1 := \; a_{00} \longrightarrow a_{01} \;\cup\; \begin{array}{c} a_{01} \\ \downarrow \\ a_{11} \end{array}$$

   Thus $\square^1 \cong \Lambda_1^2$.

4. The Hasse diagram of $\mathcal{C}\mathrm{pt}^2$ is as follows:

$$\begin{array}{ccccc} a_{00} & \longrightarrow & a_{01} & \longrightarrow & a_{02} \\ & & \downarrow & & \downarrow \\ & & a_{11} & \longrightarrow & a_{12} \\ & & & & \downarrow \\ & & & & a_{22} \end{array}$$

Thus $\mathcal{C}\mathrm{pt}^2 \cong \Delta^4 \coprod_{\Delta^1} \Delta^4$.

5. The Hasse diagram of $\square^2$ is as follows:

$$\square^2 \cong \; a_{00} \longrightarrow a_{01} \longrightarrow a_{02} \;\cup\; \begin{array}{ccc} a_{01} & \longrightarrow & a_{02} \\ \downarrow & & \downarrow \\ a_{11} & \longrightarrow & a_{12} \end{array} \;\cup\; \begin{array}{c} a_{02} \\ \downarrow \\ a_{12} \\ \downarrow \\ a_{22} \end{array}$$

Proposition D.1.13. *1.* $\square^n \cong \cup_{i=0}^n \square_i^n$ *where* $\square_i^n := \mathrm{N}(\mathrm{Cpt}^n_{(0,i)//(i,n)})$.

2. The inclusion $\square^n \subset \mathcal{C}\mathrm{pt}^n$ *is inner anodyne ([LZ12, Lemma 4.2]).*

We now define the simplicial set of compactifications.

Definition D.1.14. [LZ12, Definition 4.3] Let C be an ordinary category and $\mathcal{E}_1, \mathcal{E}_2$ be set of edges as in the condition of the Theorem D.1.3. Let $\tau : [n] \to C$ be a map. A *compactification* of τ is a map

$$\sigma : \mathrm{Cpt}^n \to C$$

such that

1. σ carries "vertical morphisms" $(i,j) \to (i',j)$ to $\mathcal{E}_1$ and "horizontal morphisms" $(i,j) \to (i,j')$ to $\mathcal{E}_2$.

2. The composition :

$$[n] \to \mathrm{Cpt}^n \xrightarrow{\sigma} C$$

is the map τ. Here the map $[n] \to \mathrm{Cpt}^n$ is the map sending $i \to (i,i)$.

Definition D.1.15. Let $\alpha = 1$ or 2, given a functor $[n] \to C$, we define the category of compactifications of τ, denoted by $\mathrm{Kpt}^\alpha(\tau)$ as follows:

1. Objects are compactifications of τ.

2. Morphisms between σ and σ' are natural transformations of functors such that the morphism $\sigma(i,j) \to \sigma'(i,j)$ is in $\mathcal{E}_\alpha$.

Definition D.1.16. (Simpler version of [LZ12, Definition 4.4]) Let C be a category with collection of edges $\mathcal{E}_1, \mathcal{E}_2$ as conditions in the theorem.Let $\tau : [n] \to C$ be a map and $\alpha = 1$ or 2. We define the *simplicial set of compactifications* to be the ∞-category $\mathcal{K}\mathrm{pt}^\alpha(\tau) := \mathrm{N}(\mathrm{Kpt}^\alpha(\tau))$.

Remark D.1.17. Some remarks on the simplicial set of compactifications are as follows:

1. A compactification $\sigma : [n] \to C$ is equivalent to a morphism of multi-marked simplicial sets:
$$\phi(\sigma) : \delta^*_{2+}\mathcal{K}\mathrm{pt}^n \to (N(C), \mathcal{E}_1, \mathcal{E}_2).$$

2. Given a compactification $\sigma : [n] \to C$, by the previous remark we get a map
$$\sigma_c : \delta^*_{2+}\mathcal{K}\mathrm{pt}^n \to (N(C), \mathcal{E}_1, \mathcal{E}_2)$$
Applying δ^{2+}_*, we have the following chain of maps:
$$\mathcal{K}\mathrm{pt}^n \to \delta^{2+}_*\delta^*_{2+}\mathcal{K}\mathrm{pt}^n \xrightarrow{\sigma_c} N(C)_{\mathcal{E}_1,\mathcal{E}_2}.$$
Applying δ^*_2, we get the map:
$$\varphi(\sigma) : \square^n \to \delta^*_2 N(C)_{\mathcal{E}_1,\mathcal{E}_2}.$$

3. The above discussion induces a map of simplicial sets:
$$\varphi : \mathcal{K}\mathrm{pt}^\alpha(\tau) \to \mathrm{Fun}(\square^n, \delta^*_2 N(C)_{\mathcal{E}_1,\mathcal{E}_2})$$

4. Let us give an explain the map $\varphi(\sigma)$ in the case of Cpt^1 and $\square^1$.
Let τ be the morphism $t y x \to y$ in C. A compactification of τ is a factorization of t of the form:
$$x \xrightarrow{t_1} x' \xrightarrow{t_2} y$$
where t_1 is in $\mathcal{E}_1$ and t_2 is in $\mathcal{E}_2$ and $t = t_2 \circ t_1$.
$\square^1$ is the union of two edges t_1 and t_2.

The map $\varphi(\sigma)$ sends the edge t_1 to a 1 simplex in $\delta^*_2 N(C)_{\mathcal{E}}$, which is of the form:
$$\begin{array}{ccc} x & \xrightarrow{\mathrm{id}} & x \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle t_1} \\ x' & \xrightarrow{\mathrm{id}} & x' \end{array}$$
and t_2 to the square
$$\begin{array}{ccc} x' & \xrightarrow{t_2} & y \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ x' & \xrightarrow{t_2} & y. \end{array}$$

5. $\square^1 \subset \mathrm{Cpt}^1$ is inner anodyne as it is the inclusion $\Lambda^2_1 \subset \Delta^2$.

Thus, given any functor $g : \delta_2^* N(C)_{\mathcal{E}} \to \mathcal{D}$ where $\mathcal{D}$ is an ∞-category and given any one simplex $\Delta^1 \to N(C)$, one can choose an arbitrary compactification of it $\sigma : \mathcal{C}\mathrm{pt}^1 \to N(C)$. This induces the map by the previous remark

$$\varphi(\sigma) : \square^1 \to \delta_2^* N(C)_{\mathcal{E}}$$

Composing with g, we get a map: $\square^1 \to \mathcal{D}$. Being an inner anodyne, we get the following dotted arrow

$$\begin{array}{ccc} \square^1 & \xrightarrow{g\circ\varphi(\sigma)} & \mathcal{D} \\ \big\downarrow & \nearrow & \\ \mathcal{C}\mathrm{pt}^1 & & \end{array}$$

By the inclusion $\Delta^1 \hookrightarrow \mathcal{C}\mathrm{pt}^1$, we get the morphism $\Delta^1 \to \mathcal{D}$. Thus we associated to a 1-simplex of $N(C)$, a 1-simplex of $\mathcal{D}$ upto a choice of compactification.

The following proposition says that the collection of compactifications in contractible.

Proposition D.1.18. *[LZ12, Theorem 4.21] The ∞-category $\mathcal{K}\mathrm{pt}^1(\tau)$ is weakly contractible.*

Remark D.1.19. The idea of the proof is to show that the underlying category is filtered. Morever given any two compactifications $\sigma, \sigma' : \mathrm{Cpt}^1 \to C$, we have a compactification $\sigma'' : \mathrm{Cpt}^1 \to C$ with morphisms $\sigma'' \to \sigma$ and $\sigma'' \to \sigma'$.

These define the necessary tools for the compactification case. We now move to define the combinatorial simplicial sets needed for proving that the map p_{cart} is a categorical equivalence. We at first define the notion of up-sets which shall lead us to the simplicial set of carteisanizations, an analogue of $\mathcal{K}\mathrm{pt}^{\alpha}(\tau)$.

Definition D.1.20. [LZ12, Definition 5.9] Let P be a partially ordered set. $Q \subset P$ is said to be an *up-set* if for every $q \in Q$ and $p \geq q$ in P implies $p \in Q$. We shall denote the category of up-sets of P by $\mathcal{U}(P)$. It is a partially ordered set where the ordering is given by inverse inclusion.

Notation D.1.21. For any partially ordered set, we denote the products (infima) by $\wedge$ and coproducts (suprema) by $\vee$. In $\mathcal{U}(P)$, we have $Q \wedge Q' = Q \cup Q'$ and $Q \vee Q' = Q \cap Q'$.

There is a canonical order preserving map $\sigma^P : P \to \mathcal{U}(P)$ defined by $p \to P_{p/}$.

There are special squares one considers in a partially ordered set, namely exact squares.

Definition D.1.22. A square in a partially ordered set is an *exact square* if it is both pushout and a pullback square.

The

Lemma D.1.23. *[LZ12, Lemma 5.18] Let $\mathcal{C}$ be an ∞-category and $F : N(\mathcal{U}(P)) \to \mathcal{C}$ be a functor. Then if F is a right Kan extension along σ^P, it sends exact squares to pullback squares.*

We move on to defining the main simplicial set Cart^n which encodes the information of how to construct cartesian squares out of commutative squares.

Definition D.1.24. Consider $[n] \times [n]$. We shall denoted the partially ordered set of non-empty up-sets of $[n] \times [n]$ by Cart^n.
We denote $\sigma^n := \sigma^{[n]\times[n]} : [n] \to [n] \to \mathrm{Cart}^n$ to be the usual map sending $(p, q) \to ([n] \times [n])_{(p,q)/}$.
Let $\mathcal{C}\mathrm{art}^n := \mathrm{N}(\mathrm{Cart}^n)$ and $\sigma^n : \Delta^n \times \Delta^n \to \mathcal{C}\mathrm{art}^n$ be the map induced from σ^n.

Remark D.1.25. Some remarks on $\mathcal{C}\mathrm{art}^n$ are as follows:

1. The diagram of $\mathcal{C}\mathrm{art}^1$ is as follows:

$$\begin{array}{ccccc} b_{00} & & & & \\ & \searrow & & & \\ & & P & \longrightarrow & b_{01} \\ & & \downarrow & & \downarrow \\ & & b_{10} & \longrightarrow & b_{11} \end{array}$$

   Here $b_{ij} := ([1] \times [1])_{(i,j)/}$ and $P = b_{01} \wedge b_{10}$.

An n-simplex of $\delta_2^* \mathrm{N}(C)_{\mathcal{E}}$ is a map $\tau : \Delta^n \times \Delta^n \to \mathrm{N}(C)$.

Definition D.1.26. Let $\tau : \Delta^n \times \Delta^n \to \mathrm{N}(C)$ be a map. We define the simplicial set $\mathcal{K}\mathrm{art}(\tau)$ which is defined as the pullback of the diagram:

$$\begin{array}{ccc} & & \mathcal{K}\mathrm{art}(\tau)_{\mathrm{RKE}} \\ & & \downarrow \\ \Delta^0 & \xrightarrow{\tau} & \mathrm{Fun}(\Delta^n \times \Delta^n, \mathrm{N}(C)) \end{array}$$

where $\mathcal{K}\mathrm{art}(\tau)_{\mathrm{RKE}}$ is the sub-simplicial set of $\mathrm{Fun}(\mathcal{C}\mathrm{art}^n, \mathrm{N}(C))$ which are right Kan extensions along σ^n.

Proposition D.1.27. *[LZ12, Remark 5.22] If C admits pullbacks, the simplicial set $\mathcal{K}\mathrm{art}(\tau)$ is a contractible Kan complex.*

We need to give a marked structure on the simplicial sets $\mathcal{C}\mathrm{art}^n$. For this, we need some more notations and maps in the simplicial sets $\mathcal{C}\mathrm{art}^n$.

Notation D.1.28. [LZ12, Notation 5.23]

1. We have a map:
$$\pi^n : \mathrm{Cart}^n \to [n] \times [n]$$
   defined as:
$$\pi^n(P) := (\min_{(p,q)\in P} p, \min_{(p,q)\in P} q).$$

2. $\pi^n \circ \sigma^n = \mathrm{id}_{[n]\times[n]}$.

3. Other than σ^n, we have two maps:

$$\xi^n, \eta^n : [n] \times [n] \to \mathrm{Cart}^n$$

defined by

$$\xi^n(p,q) := \sigma^n(p,0) \wedge \sigma^n(0,q); \eta^n(p,q) := \sigma^n(p,n) \wedge \sigma^n(n,q).$$

4. For $(p,q) \in [n] \times [n]$, we denote

$$\boxplus^n_{(p,q)} := N(\mathrm{Cart}^n_{\xi^n(p,q)//\eta^n(p,q)})$$

5. Denote

$$\boxplus^n := \cup_{(p,q)\in[n]\times[n]} \boxplus^n_{(p,q)}\,.$$

Remark D.1.29. Some remarks on the above notations:

1. The functors ξ^n and η^n satisfy the following property:

$$\xi^n(p,q) \leq \sigma^n(p,q) \leq \eta^n(p,q)$$

2. The definition of $\boxplus^n$ is analog to the definition of $\square^n$. In the case of $\square^n$, we have the functors: $p, q : [n] \to \mathrm{Cpt}^n$ defined as $p(i) = (0,i)$ and $q(i) = (i,n)$. And $p(i) \leq (i,i) \leq q(i)$. The functors p and q are analog to the functors σ^n, η^n which motivates defining $\boxplus^n_{(p,q)}$ and $\boxplus^n$ in the similar way one defined $\square^n_i$ and $\square^n$.

Definition D.1.30. [LZ12, Notation 5.25] Consider the bi-marked simplicial set $(\Delta^n \times \Delta^n, \mathcal{F}'_1 := (\epsilon^2_1\Delta^{n,n})_1, \mathcal{F}'_2 := (\epsilon^2_2\Delta^{n,n})_1)$. Let $(\mathcal{C}\mathrm{art}^n, \mathcal{F})$ be the marked-simplicial set $(\mathcal{C}\mathrm{art}^n, \mathcal{F}_1 := (\pi^n)^{-1}(\mathcal{F}'_1), \mathcal{F}_2 := (\pi^n)^{-1}(\mathcal{F}'_2))$.
We define $(\mathcal{C}\mathrm{art}^n, \mathcal{F}^{\mathrm{cart}})$ to the 2-tiled simplicial set where the 2-tiling is given by $\mathcal{F}_{12} := \mathcal{F}_1 \star^{\mathrm{cart}} \mathcal{F}_2$.

Remark D.1.31. 1. We have a natural inclusion of simplicial sets:

$$i_c : \delta_2^*\delta_*^{2\square}(\mathcal{C}\mathrm{art}^n, \mathcal{F}^{\mathrm{cart}}) \hookrightarrow \delta_2^*\delta_*^2(\mathcal{C}\mathrm{art}^n, \mathcal{F}).$$

This is because the m-simplices of $\delta_2^*\delta_*^{2\square}(\mathcal{C}\mathrm{art}^n, \mathcal{F}^{\mathrm{cart}})$ are maps $\Delta^m \times \Delta^m \to \mathcal{C}\mathrm{art}^n$ such that for every map:

$$\Delta^1 \times \Delta^1 \to \Delta^m \times \Delta^m \to \mathcal{C}\mathrm{art}^n$$

is in $\mathcal{F}_{12}$.

2. We have maps

$$\psi : \mathcal{K}\mathrm{art}(\tau) \to \mathrm{Fun}(\delta_2^*\delta_*^2(\mathcal{C}\mathrm{art}^n, \mathcal{F}), \delta_2^* N(C)_{\mathcal{E}})$$

and

$$\psi_c : \mathrm{Kart}(\tau) \to \mathrm{Fun}(\delta_2^*\delta_*^{2\square}(\mathcal{C}\mathrm{art}^n, \mathcal{F}^{\mathrm{cart}}, \delta_2^* N(C)^{\mathrm{cart}}_{\mathcal{E}}).$$

These maps are induced from an extension of the map τ to a map $h : \mathcal{C}\mathrm{art}^n \to N(C)$. The map h of multi-marked simplicial sets and multi-tiled simplicial sets

$$(\mathcal{C}\mathrm{art}^n, \mathcal{F}) \to (N(C), \mathcal{E})$$

and

$$(\mathcal{C}\mathrm{art}^n, \mathcal{F}^{\mathrm{cart}}) \to (\mathsf{N}(\mathsf{C}), \mathcal{E}, \mathcal{E}_{12}).$$

The maps ψ and ψ_c are constructed by applying functors δ_2^*, δ_*^2 and $\delta_*^{2\square}$ on the above two maps.

3. It is not clear that the inclusion map i_c is an inner anodyne unlike the inclusion of $\square^n \subset \mathcal{C}\mathrm{pt}^n$. Thus, we need to have a map of simplicial sets $i_{XY} : X \to Y$ which is a cofibration and there exists a commutative square of the form

$$\begin{array}{ccc} X & \xrightarrow{q_{X_c}} & \delta_2^*\delta_*^{2\square}(\mathcal{C}\mathrm{art}^n, \mathcal{F}^{\mathrm{cart}}) \\ \downarrow{\scriptstyle i_{XY}} & & \downarrow \\ Y & \xrightarrow{q_Y} & \delta_2^*\delta_*^2(\mathcal{C}\mathrm{art}^n, \mathcal{F}). \end{array}$$

We have the following lemma:

Lemma D.1.32. *[LZ12, Lemma 5.35] There exists simplicial sets* $B^n \xrightarrow{i} A^n$ *such that:*

1. $A^n = (\Delta^n \times \Delta^n)^\# \times (\mathcal{C}\mathrm{art}^n)^\flat$.

2. $B^n = \bigcup_{x \le y} \mathsf{N}(I_{x,y})^\# \times (\mathcal{C}\mathrm{art}^n_{x//y})^\flat \xrightarrow{i} A^n$. *Here* $I_{x,y} \subset [n] \times [n]$ *consisting of pairs* (x, y) *such that* $\xi^n(p, q) \le x \le y \le \eta^n(p, q)$.

3. i *is a cofibration and there is a commutative diagram*

$$\begin{array}{ccc} B^n & \xrightarrow{\beta_n} & \delta_2^*\delta_*^{2\square}(\mathcal{C}\mathrm{art}^n, \mathcal{F}^{\mathrm{cart}}) \\ \downarrow{\scriptstyle i} & & \downarrow \\ A^n & \xrightarrow{\alpha_n} & \delta_2^*\delta_*^2(\mathcal{C}\mathrm{art}^n, \mathcal{F}). \end{array}$$

D.1.4 Proving p_{comm} is a categorical equivalence.

We try to explain the key idea of the proving that p_{comm} is a categorical equivalence. A part of the proof uses the idea of category of simplices over a simplicial set. This is discussed in detail in [LZ12, Section 2].

Let $Y = \delta_2^*\mathsf{N}(\mathsf{C})_{P,O}$ and $Z = \mathsf{N}(\mathsf{C})$. We apply Lemma D.1.7 in our setting. Consider the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\nu} & \mathrm{Fun}(\Delta^l, \mathcal{D}) \\ \downarrow{\scriptstyle p_{\mathrm{comm}}} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{u} & \mathrm{Fun}(\partial\Delta^l, \mathcal{D}) \end{array} \tag{D.11}$$

where $l = 0, 1, 2$. We need to construct a map $u : Z \to \mathrm{Fun}(\Delta^l, \mathcal{D})$. Let $\tau : \Delta^n \to Z$. We need to define $u(\tau) : \Delta^n \to \mathrm{Fun}(\Delta^l, \mathcal{D})$.
We have the following commutative diagram:

$$
\begin{array}{ccccccc}
N(\tau) & \xrightarrow{h'} & \mathrm{Fun}(\mathcal{C}\mathrm{pt}^n \times \Delta^l, \mathcal{D}) & & \xrightarrow{\mathrm{res}_2} & & \mathrm{Fun}(\Delta^n \times \Delta^l, \mathcal{D}) \\
\downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} & & & & \downarrow{\scriptstyle \mathrm{id} \times p} \\
\mathcal{K}\mathrm{pt}^{\alpha}(\tau) & \xrightarrow{h} & \mathrm{Fun}(H, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\partial\Delta^l \times \mathcal{C}\mathrm{pt}^n, \mathcal{D}) & \xrightarrow{\mathrm{res}_1} & \mathrm{Fun}(\partial\Delta^l \times \Delta^n, \mathcal{D})
\end{array}
\tag{D.12}
$$

where:

1. $\mathrm{res}_2, \mathrm{res}_1$ is induced by inclusion $\Delta^n \subset \mathcal{C}\mathrm{pt}^n$.
2. $H = \Delta^l \times \square^n \coprod_{\partial\Delta^l \times \square^n} \partial\Delta^l \times \mathcal{C}\mathrm{pt}^n$ and j is induced by the inclusion $H \hookrightarrow \Delta^l \times \mathcal{C}\mathrm{pt}^n$.
3. $h : \mathcal{K}\mathrm{pt}^{\alpha}(\tau) \to H$ be the map induced by $\varphi(\tau), \nu$ and w.
4. $N(\tau)$ is the simplicial set such that the leftmost square is pullback.

We have the following statements:

1. j is trivial Kan fibration. This follows from [Lur09, Corollary 2.3.2.5] applied to the inner anodyne $i : \square^n \subset \mathcal{C}\mathrm{pt}^n$ and the inner fibration $\mathcal{D} \to \Delta^0$.
2. This implies that j' is a trivial Kan fibration.
3. As $\mathcal{K}\mathrm{pt}^{\alpha}(\tau)$ is weakly contractible and j' is a weak homotopy equivalence implies $N(\tau)$ is weakly contractible.

Let us denote the composition $\mathrm{res}_2 \circ h'$ by ϕ_n. The rest of the argument uses the notion of category of simplices. Indeed the collection ϕ_n induces a natural transformation

$$\phi : N \to \mathrm{Map}[Z, \mathrm{Fun}(\Delta^l, \mathcal{D})]$$

between functors over category of simplices over Z (see [LZ12, Notation 2.6] for further details). The above commutative diagram also implies that morphism ϕ when mapped to $\mathrm{Map}[Z, \mathrm{Fun}(\partial\Delta^l, \mathcal{D})]$ is constant i.e. the image of ϕ in $\mathrm{Map}[Z, \mathrm{Fun}(\partial\Delta^l, \mathcal{D})]$ is a map $u : Z \to \mathrm{Fun}(\partial\Delta^l, \mathcal{D})$.
As the morphism $\mathrm{Map}[Z, \mathrm{Fun}(\Delta^l, \mathcal{D})] \to \mathrm{Map}[Z, \mathrm{Fun}(\partial\Delta^l, \mathcal{D})]$ is an injective fibration with respect to the injective model structure on $\mathrm{Fun}(\Delta_{/Z}, \mathrm{Set}_{\Delta})$ (see [LZ12, Proposition 2.8] for more details), it satisfies right lifting property with respect to anodyne maps. As $N(\sigma)$ is weakly contractible, then inclusion $N \hookrightarrow N^{\triangleleft}$ is an anodyne map. Thus the map ϕ admits an extension

$$\phi' : N^{\triangleleft} \to \mathrm{Map}[Z, \mathrm{Fun}(\Delta^l, \mathcal{D})].$$

Evaluating ϕ' at the cone point gives us a morphism $w : Z \to \mathrm{Fun}(\Delta^l, \mathcal{D})$ such that $p \circ w = u$ and $w \circ p_{comm}$,ν are homotopic relative to $\mathrm{Fun}(\partial\Delta^l, \mathcal{D})$.

D.2 Partial adjoints.

Theorem D.2.1. *Consider quadruples* (I, J, R, f) *where* $J \subset I$ *are finite sets,* R *an* I*-simplicial set and* $f : \delta_I^* R \to \mathrm{Cat}_\infty$ *a functor satisfying the following conditions:*

1. *For every* $j \in J$ *and for every edge* $e \in \epsilon_j^I(R)$*,* $f(e)$ *has a right adjoint.*

2. *For every* $i \in J^c := I/J$ *and* $j \in J$*, every square* $\tau \in (\epsilon_{i,j}^I R)_{1,1}$ *is right adjointable.*

Then, there exists a functor $f_J : \delta_{I,J}^* R \to \mathrm{Cat}_\infty$ *satisfying the following conditions:*

1. f *and* f_J *are the same functors on the sub-simplicial set* $\delta_{J^c}^*(\Delta_{i_c})_* R \subset \delta_I^* R, \delta_{I,J}^* R$*. Here* $i_c : J^c \subset I$*.*

2. *For every* $j \in J$ *and for every edge* $e \in \epsilon_j^I(R)$*,* $f_J(e)$ *is a right adjoint to* $f(e)$*.*

3. *For every* $i \in J^c, j \in J$ *any every square* $\tau \in (\epsilon_{i,j}^I R)_{1,1}$*,* $f_J(\tau)$ *is right adjoint to* $f(\tau)$*.*

Remark D.2.2. The theorem of partial adjoints help us to encode the notion of smooth and proper base change in the setting of six operations. Let R' be the bi marked simplicial set $(N(\mathrm{Sch}), \mathrm{ALL}, \mathrm{open})$ where the collection ALL considers all morphisms and the collection open considers only open immersions. Let

$$R := \delta_*^2 R'$$

and let f be the functor

$$f : \delta_2^* R \to \mathrm{Cat}_\infty$$

which on the level of zero simplices takes $X \to \mathcal{D}^\otimes(X)$ and for a morphism $h : X \to Y$ sends it to the pullback h^*(here we follow the notation defined in Notation 4.3.1). Then the theorem along with the conditions of Notation 4.3.1 allows us to define a new map

$$f' : \delta_{2,\{2\}}^* R \to \mathrm{Cat}_\infty$$

which now sends an open immersion $h : X \to Y$ to $h_\#$.
A similar statement holds for proper maps.

D.3 Construction of the enhanced operation map.

We review the construction of the enhanced operation map associated to a functor $\mathcal{D}$ which satisfies the conditions of Notation 4.3.1. This is explained in [Rob14, Section 9.4.13].

Let $\mathrm{Fun}(\Delta^1, \mathrm{Sch}_{fd})$ be the functor category of base schemes. Let

$$\begin{array}{ccc} Y_0 & \xrightarrow{u} & Y_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\ X_0 & \xrightarrow{v} & X_1 \end{array} \tag{D.13}$$

be an edge in $\mathrm{Fun}(\Delta^1, \mathrm{Sch}_{fd})$. We denote the following:

1. $F :=$ all such squares such that u and v are separated morphisms of finite type.
2. $P :=$ all such squares such that u and v are proper.
3. $O :=$ all such squares such that u and v are open.
4. ALL := all such squares such that u and v can be any morphism of schemes.

Consider the following composition:

$$\begin{array}{c} \delta^*_{3,\{1,2,3\}}((\mathrm{Fun}(\Delta^1, \mathrm{Sch}_{fd}))_{P,O,ALL})^{cart} \\ \downarrow \\ \delta^*_{2,\{1,2\}}((\mathrm{Fun}(\Delta^1, \mathrm{Sch}_{fd}))_{F,ALL'})^{cart} \\ \downarrow \\ \delta^*_{1,\{1\}}(\mathrm{Fun}(\Delta^1, \mathrm{Sch}_{fd})) = \mathrm{Fun}(\Delta^1, \mathrm{Sch}_{fd})^{op} \\ \downarrow \\ \mathrm{Mod}(\mathrm{Pr}^L_{stb}). \end{array} \tag{D.14}$$

Using the definition of module objects in terms of the operad $\mathrm{Pf}^{\otimes}$, the above composition can be formulated as a morphism

$$\mathcal{D}_{1,2,3} : \delta^*_{4,\{1,2,3\}}(((\mathrm{Fun}(\Delta^1, \mathrm{Sch}_{fd}))_{P,O,ALL})^{cart} \boxtimes \mathrm{Pf}^{\times}) \to \widehat{\mathrm{Cat}_{\infty}}.$$

Here $- \boxtimes -$ denotes the exterior product of multisimplicial sets ([LZ12, Definition 3.7]).
We shall use the theorem of partial adjoints Theorem D.2.1 repeatedly to construct the enhanced operation map. It is constructed in the following steps.

1. We use Theorem D.2.1 on the functor $\mathcal{D}_{1,2,3}$ and on the simplicial set $R = \mathrm{op}^4_{\{1,2,3\}}(\mathrm{Sch}_{fd})^{cart}_{P,O,ALL}$ and $J = \{1\} \subset I = \{1, 2, 3, 4\}$. The theorem can be applied because of the following:
   (a) Condition (1) of Theorem D.2.1 exists because of existence of f_*.
   (b) Condition (2) of Theorem D.2.1 exists because of $2(a)$ and $2(b)$ of Notation 4.3.1.

   This yields us the functor

$$\mathcal{D}_{2,3} : \delta^*_{4,\{2,3\}}((\mathrm{Sch}_{fd})^{cart}_{P,O,ALL} \boxtimes \mathrm{Pf}^{\otimes}) \to \widehat{\mathrm{Cat}_{\infty}}. \tag{D.15}$$

2. We apply the dual version of Theorem D.2.1 to $\mathcal{D}_{2,3}$ where $R = \mathrm{op}^3_{\{2,3\}}((\mathrm{Sch}_{fd})^{cart}_{P,O,ALL} \boxtimes \mathrm{Pf}^{\otimes})$ and $J = \{2\} \subset I = \{1, 2, 3\}$. The conditions of the theorem are satisfied because:
   (a) Condition 1 of Theorem D.2.1 exists because of the existence of $f_{\#}$.
   (b) Condition 2 of Theorem D.2.1 exists because of 3 and $1(a), (b)$ of Notation 4.3.1.

This yields us the functor:

$$\mathcal{D}_3 : \delta^*_{4,\{3\}}((\mathrm{Sch}_{\mathtt{fd}})^{\mathrm{cart}}_{\mathrm{P,O,ALL}} \boxtimes \mathrm{Pf}^{\otimes}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathtt{stb}}) \tag{D.16}$$

The map $\mathcal{D}_3$ now can be formulated as a morphism

$$\mathcal{D}_3 : \delta^*_{3,\{3\}}(\mathrm{Sch}_{\mathtt{fd}})^{\mathrm{cart}}_{\mathrm{P,O,ALL}} \to \mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathtt{stb}}).$$

3. Now we use the compactification theorem (Theorem D.1.3) on $\mathcal{D}_3$. This can be applied as we have the compactification for separated and finite type morphisms. The categorical equivalence yields a functor

$$\mathrm{EO}(\mathcal{D}^{\otimes}) : \delta^*_{2,\{2\}}\,\mathrm{Fun}(\Delta^1, \mathrm{Sch}_{\mathtt{fd}})^{\mathrm{cart}}_{\mathrm{F,ALL}} \to \mathrm{Mod}(\mathrm{Pr}^{\mathrm{L}}_{\mathtt{stb}}) \tag{D.17}$$

We call the map $\mathrm{EO}(\mathcal{D}^{\otimes})$ as the *enhanced operation map*.

BIBLIOGRAPHY

[Ayo07a] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. *Astérisque*, (314):x+466 pp. (2008), 2007.

[Ayo07b] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II. *Astérisque*, (315):vi+364 pp. (2008), 2007.

[BS15] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. *Astérisque*, (369):99–201, 2015.

[CD19] Denis-Charles Cisinski and Frédéric Déglise. Triangulated categories of mixed motives. *Springer Monographs in Mathematics*, 2019.

[EG98] Dan Edidin and William Graham. Equivariant intersection theory. *Invent. Math.*, 131(3):595–634, 1998.

[FHT11] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Loop groups and twisted k-theory i. *Journal of Topology*, 4(4):737–798, December 2011.

[FP97] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.

[Ful84] William Fulton. *Intersection Theory*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1984.

[Gro17] Philipp Gross. Tensor generators on schemes and stacks. *Algebr. Geom.*, 4(4):501–522, 2017.

[HL10] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves*. Cambridge Mathematical Library. Cambridge University Press, 2 edition, 2010.

[Hoy17] Marc Hoyois. The six operations in equivariant motivic homotopy theory. *Adv. Math.*, 305:197–279, 2017.

[Knu71] Donald Knutson. *Algebraic spaces*, volume 203. Springer, Cham, 1971.

[Lan71] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer New York, New York, NY, 1971.

[Lan21] Markus Land. *Introduction of infinity-categories*. Birkhäuser, Cham, 2021.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2000.

[LO08a] Yves Laszlo and Martin Olsson. The six operations for sheaves on Artin stacks. I. Finite coefficients. *Publ. Math. Inst. Hautes Études Sci.*, (107):109–168, 2008.

[LO08b] Yves Laszlo and Martin Olsson. The six operations for sheaves on Artin stacks. II. Adic coefficients. *Publ. Math. Inst. Hautes Études Sci.*, (107):169–210, 2008.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur17] Jacob Lurie. Higher algebra. http://people.math.harvard.edu/~lurie/papers/HA.pdf, 2017.

[Lur18a] Jacob Lurie. Kerodon. https://kerodon.net, 2018.

[Lur18b] Jacob Lurie. Spectral algebraic geometry. https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf, 2018.

[LZ12] Yifeng Liu and Weizhe Zheng. Gluing restricted nerves of ∞-categories. *arXiv e-prints*, page arXiv:1211.5294, November 2012.

[LZ17] Yifeng Liu and Weizhe Zheng. Enhanced six operations and base change theorem for higher artin stacks. 2017.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[Ols05] Martin Olsson. Sheaves on artin stacks. https://math.berkeley.edu/~molsson/qcohrevised.pdf, 2005.

[Rez] Charles Rezk. Stuff about quasicategories. https://faculty.math.illinois.edu/~rezk/quasicats.pdf.

[Rob14] Marco Robalo. Théorie homotopique motivique des espaces noncommutatifs. https://webusers.imj-prg.fr/~marco.robalo/these.pdf, 2014.

[Rob15] Marco Robalo. K-theory and the bridge from motives to noncommutative motives. *Adv. Math.*, 269:399–550, 2015.

[Sta21] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2021.

[Tot99] Burt Totaro. *The Chow ring of a classifying space*, volume 67 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1999.

[Tot04] Burt Totaro. The resolution property for schemes and stacks. *J. Reine Angew. Math.*, 577:1–22, 2004.