Stable ∞-categories and Structured Ring Spectra

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1 Intro

In the second to last talk we saw those strange objects called spectra. They are roughly speaking a sequence of spaces with some relation between the various pieces. We saw that those objects are interesting (among many other properties) because they represents (extraordinary) cohomology theories, like our beloved K-theory, complex cobordism, ecc. We called their homotopy category the "Stable Homotopy Category". In a certain sense this *stability* is a kind of abelian property of the category: we have some objects called fiber and cofiber instead of kernels and cokernels and those objects will behave well. Moreover the homotopy category of those stable categories is a triangulated ordinary category. As Enzo mentioned last time: to have a(n ∞-) category that remembers (all) the (higher) homotopies is useful and solve some problems of *higher coherences*, like *"gluing of derived categories"*.

Beware that I will be very imprecise in some points in order to stress the idea behind what I'll try to expose and to avoid issues that maybe for people that encounter these things for the first time could be redundant and confusing.

I would advise to keep in mind during this talk the prototypical example of stable infinity category: the derived category of a ring $D_\infty(R)$. The usual derived category $D(R)$ that we are used to consider is only a triangulated category where we forget how homotopies behave while $D_\infty(R)$ is an ∞-category that help us to keep track of all homotopies (also higher homotopies, i.e. homotopies between homotopies) and whose homotopy category $\mathrm{h}(D_\infty(R)) = D(R)$ is the usual triangulated category. Now I will use some terminology written in italic that we'll see later on the talk: it happens that this derived ∞-category is *stable*. Indeed we now that $D(R) = Ch(R)$, i.e. the category of chain complexes of R-modules, that can be recovered as a *stabilization* of R-modules since the *suspension* and the *desuspension* is made by (± 1)-shift of complexes. I abused some notation because $Ch(R)$ is not a simplicial set (in particular not an ∞-category), I should take the nerve of this 1-category to be more precise $D_\infty(R) = N(Ch(R))$.

References for this talk are the book "Higher Algebra" of Lurie as Chirantan mentioned (paragraph 1.1 and 1.4) and the very good introduction of Gepner called "Introduction to Higher Categorical Algebra".

2 Stable ∞-Categories

Definition 2.1. Let $\mathcal{C}$ and ∞-category, a **zero object** is an object of $\mathcal{C}$ such that it is both initial and final. If $\mathcal{C}$ has a zero object then we call it a **pointed ∞-category**.

Definition 2.2. Let $\mathcal{C}$ be a pointed ∞-category and consider the following object that we'll call a **triangle**:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

If it is a pullback (pushoput) square we call $X \longrightarrow Y \longrightarrow Z$ is a **fiber** (**cofiber**) **sequence**.

Remark 2.3. If you look carefully enough to the definition of a triangle you'll see that in our ∞-world the composition $g \circ f$ is given by a 2-simplex and that this composition is 0 up to homotopy (while in the classical world we're used to thing that are 0 on the nose). If you want more detail see the Remark 1.1.1.5 in Higher Algebra of Lurie.

Definition 2.4. An ∞-category $\mathcal{C}$ is called **stable** if:

1. it is pointed (i.e. there exists a zero object).
2. it admits all fibers and cofibers.
3. fiber and cofiber sequences are the same.

Remark 2.5. The third condition is the analogous of the classical condition for abelian categories that states that the image of a morphism is isomorphic to its coimage (i.e. we have an exact sequence).

In the context of pointed ∞-categories we can talk about suspension and loop functors referring to the (homotopy) pushout and (homotopy) pullback diagrams:

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array} \qquad\qquad \begin{array}{ccc} \Omega Y & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Y \end{array}$$

where $\Sigma X = * \coprod_X^h *$ is the suspension of X and $\Omega X = * \times_Y^h *$ is the loop space of Y (if they exists!! In the context of stable categories we don't need to worry, but we can make sense of this notions also in categories that are not stable and, spoiler, we can stabilize them).

(**Caveat:** I shouldn't say homotopy pushout/pullback but just pushout/pullback in the contest of ∞-categories since all the things that we're considering are intrinsically *homotop-ish*: the limit/colimit isn't unique, but it's a contractible space therefore unique up to homotopy!)
The stable categories can be characterised by this proposition:

Proposition 2.6. *For a pointed ∞-category $\mathcal{C}$ that admits fibers and cofibers the following conditions are equivalent:*

1. *$\mathcal{C}$ is stable.*
2. *the suspension and loop functors are one the inverse of the other.*

Moreover the stable categories as we mentioned before have a pleasant property of being almost nice as an abelian category in the ∞-categorical world, indeed we also have:

Proposition 2.7. *The homotopy category of a stable category is a triangulated category where the distinguished triangles are of the form:*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

Due to these two proposition above it's common to write $X[1] := \Sigma X$ and $X[-1] := \Omega X$.

If we have a functor between stable ∞-categories $F : \mathcal{C} \longrightarrow \mathcal{D}$ such that F send the zero object to the zero object then clearly it send triangles into triangles. It could be a good candidate to be the correct notion of functor between stable ∞-categories, but we need more (as in the usual ordinary case): we need it to be *exact*. The next proposition give equivalent definition for what we want to call exact:

Proposition 2.8. *$F : \mathcal{C} \longrightarrow \mathcal{D}$ functor between stable ∞-categories. TFAE:*

1. *F commutes with finite limits and we call it* ***left exact****.*
2. *F commutes with finite colimits and we call it* ***right exact****.*
3. *F commutes with finite limits and colimits and we call it* ***exact****.*

3 Stabilization

Quoting Lurie: one very broad goal of homotopy theory is to classify continuous maps between topological spaces up to homotopy. The set $[X, Y]$ between (pointed) topological spaces is very complicated in general. But if $X = \Sigma X'$ is a suspension of another (pointed) space then $[X, Y] \simeq \pi_1(\mathrm{Map}(X', Y))$ has a group structure. If $X' = \Sigma X''$ then $[X, Y] \simeq \pi_2(\mathrm{Map}(X'', Y))$ is abelian. Then heuristically since the map $X \longrightarrow \Sigma X$ is functorial we have:

$$[X, Y] \longrightarrow [\Sigma X, \Sigma Y] \longrightarrow [\Sigma^2 X, \Sigma^2 Y] \longrightarrow \dots$$

and we can study the groups $[\Sigma^n X, \Sigma^n Y]$ as an approximation of $[X, Y]$. Taking the colimit $\varinjlim_n [\Sigma^n X, \Sigma^n Y]$ we recover the *stable mapping space* (in the case where $X = \mathbb{S}^n$ we get the stable n^{th}-homotopy group that we saw with Gürkan). Why do we want to study these objects? We want to study them because they are a sort of linearization of $[X, Y]$, so they are easier to study and give us some information about our starting object $[X, Y]$. So basically we prefer to study homotopy classes of *stable maps* instead of homotopy classes of maps.

We saw with Gürkan the ordinary *stable homotopy category*, but is there an ∞-analogous? Of course there is! The basic idea behind the theory of ∞-categories is to make the Hom-sets become *mapping spaces* (i.e. somehow real topological spaces!), so we replaces **Set** with the ∞-category of spaces $\mathcal{S}$ that could be thought as the (coherent) nerve of the category of (small) Kan complexes. So we get as a starting point an ∞-version of the category of spaces, then as in the classical case we can begin from there to construct spectra. There are several ways of doing it, one of them is to use the characterization of stable categories given in Proposition 2.6: so we want to formally invert the suspension/loop functor. To do that we take the pointed category of spaces $\mathcal{S}_*$ and we take the inverse limit of the tower:

$$\mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \dots$$

and we call the resulting object $\mathcal{Spt}$ the stable ∞-category of spectra (in other words we described it as the category of infinite loop spaces). It's homotopy category is exactly the triangulated category that we saw in Gürkan's talk: the world is safe!

Remark 3.1 (**Shadows of Higher Algebra**). Notice that given a spectrum $\{E(n)\}_{n\in\mathbb{Z}}$ we have in particular that $E(0) \simeq \Omega E(1)$, so there is a multiplication $E(0) \times E(0) \longrightarrow E(0)$ we defined and associative up to *coherent homotopy*. More is true: $E(0) \simeq \Omega^n E(n)$ as the structure of n^{th}-loop space. This makes $E(0)$ be a commutative monoid object in the ∞-category $\mathcal{S}$ of spaces. There is also a converse: the map $\{E(n)\} \longrightarrow E(0)$ induces an equivalence between connective spectra $\mathcal{Spt}^{\mathrm{cn}} \subseteq \mathcal{Spt}$ and grouplike (i.e. we have an invertible binary operation) commutative monoid objects in spaces $\mathcal{Mon}^{\mathrm{gp}}_{\mathrm{Comm}}(\mathcal{S})$. This strange category in ordinary contex where we replaces spaces with sets it's just the category of abelian groups, so we have "translated" in the language of ∞-categories the relation between abelian groups and sets that now is recovered by spaces and spectra. In some sense we're making some *higher algebra*: we're looking at abelian groups *up to homotopy*. This kind of heuristic can be adopted to talk about "higher" rings, modules, etc.

In general if we have a pointed ∞-category $\mathcal{C}$ that admits fibers and cofibers we can talk about suspension and desuspension $\Sigma_{\mathcal{C}}, \Omega_{\mathcal{C}}$ and basically with the same argument we can form its **stabilization** taking the inverse limit of:

$$\mathcal{C}_* \xleftarrow{\Omega_{\mathcal{C}}} \mathcal{C}_* \xleftarrow{\Omega_{\mathcal{C}}} \mathcal{C}_* \xleftarrow{\Omega_{\mathcal{C}}} \dots$$

that we'll denote $\mathcal{Spt}(\mathcal{C})$ the ∞-category of spectrum object of $\mathcal{C}$.

4 (Very Few and Sketchy Comments on) Higher Algebra

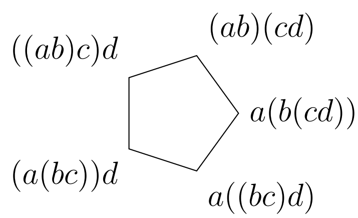
To do *higher algebra*, i.e. to talk about ring, module structure and whatever, on spectra. The basic thing to do to start is to try to clarify what a ring should be: we need a product on spectra! It's not an easy story, it was an issue for a long time: the first resolutions of this problem were the symmetric spectra of Hovey-Shipley- Smith and the S-modules of Elmendorf-Kriz-Mandell-May. We would like to do some operation level-wise (since spectra are sequence of spaces roughly speaking). The main problem is that the smash product of spectra is well defined up to homotopy: think about the derived category $D(R)$ (that could be thought as a category of spectrum objects) with the derived tensor product (that indeed is well defined only up to quasi-isomorphism)! Anyhow we can take as a definition, that the smash product of suspension spectra is given by:

$$(\Sigma^\infty X_0) \otimes (\Sigma^\infty X_1) \otimes \dots \otimes (\Sigma^\infty X_n) := \Sigma^\infty (X_0 \wedge X_1 \wedge \dots \wedge X_n)$$

By HTT 6.3.3.6, any spectrum admits a canonical presentation by desuspended suspension spectra:

$$A \simeq \mathrm{colim}_n\, \Sigma^{-n}\Sigma^{\infty} A_n$$

so we can use this and the definition above to construct a smash product for spectra in general.
Now that we have a candidate for our *ring product* we should be able to talk about (*associative, unital*) *algebra objects*. In ordinary category theory, given a monoidal category $(\mathbf{C}, \otimes, 1)$, an algebra object in $\mathbf{C}$ is given by an object A with a unit $e : 1 \longrightarrow A$ and a product $\mu : A \otimes A \longrightarrow A$ satisfying the usual associativity and unitality conditions. When passing to an *higher* context the matter is more complicated since we should deal with associativity, commutativity, ecc *up to coherent homotopy*. For example let's come back to loop spaces ΩX: there we have that the loops $(ab)c$ and $a(bc)$ are not the same (we spend different times on the loops), but they are the same up to homotopy! But then we have to check associativity also for longer sequences and for example for a, b, c, d we get the Stasheff pentagon that relates the various path by homotopies:



and so on. To keep track of this higher coherence data and actually to talk about symmetric monoidal ∞-categories we should use something called operads or at least co/Cartesian fibration of ∞-categories (see the very good notes of Gepner "An introduction to Higher Categorical Algebra"), but we don't have enough time. The idea behid *structured* ring spectra as $\mathbb{A}_\infty$-ring spectra (associative ring spectra) and $\mathbb{E}_\infty$-ring spectra (commutative ring spectra) is that we have the multiplication $\mu : E \otimes E \longrightarrow E$ that is associative/commutative up to coherent homotopy.
Let's try to see or at least to grasp a little bit more what's going on with associative ring spectra. We should believe that there is a good generalization of symmetric monoidal categories for ∞-categories (they are commutative monoids in the category of ∞-categories, where commutative monoid should be defined carefully in this contex).

Definition 4.1. We define an (ordinary) category $\mathbf{Ass}^{\otimes}_{act}$. The objects are finite sets. A morphism from S to T is given by a map $S \longrightarrow T$ together with a linear ordering on the preimages for each $t \in T$. Composition is defined by composition of maps with lexicographic ordering on preimages. The category $\mathbf{Ass}^{\otimes}_{act}$ becomes symmetric monoidal by disjoint union.

Note that $[1] \in \mathbf{Ass}^{\otimes}_{act}$ is naturally an associative algebra object.

Exercise 4.2. Given a symmetric monoidal (ordinary) category $\mathbf{C}$, there is an equivalence between algebra objects of $\mathbf{C}$ and symmetric monoidal functors $F : \mathbf{Ass}^{\otimes}_{act} \longrightarrow \mathbf{C}$. The equivalence is given by the evaluation of the functors at [1].

If we take the ***homotopy coherent nerve*** (this is what we mean when we say to take all the higher coherence) of $\mathbf{Ass}^{\otimes}_{act}$ we get a symmetric monoidal ∞-category $N(\mathbf{Ass}^{\otimes}_{act})$.

Definition 4.3. An algebra object in a symmetric monoidal ∞-category $\mathcal{C}$ is a symmetric monoidal functor $N(\mathbf{Ass}^{\otimes}_{act}) \longrightarrow \mathcal{C}$. The ∞-category $\mathcal{A}lg(\mathcal{C})$ is the ∞-category of symmetric monoidal functors.

To define commutative algebra objects in higher context we can mimic the same idea, just replacing $\mathbf{Ass}^{\otimes}_{act}$ with the category of finite sets **Fin** (basically it is the same category but we forget about the ordering of the preimages of morphisms: we are making it *commutative*).

Example 4.4. An example of commutative ring spectra, i.e. a commutative algebra object in $\mathcal{S}pt$, is given by Eilenberg-MacLane spectra. Using Dold-Kan $HA_n = DK(A[n])$ (remember HA is connective, i.e. it lives in $\mathcal{C}h_{\leq 0}$) and it is commutative *on the nose*, meaning that the commutative structure is induced by the multiplication of A as a ring, so it is on the nose and not only up to homotopy.