Baby Seminar WS 2019/20 Program: Hochschild and Cyclic Homology.

1 What is the seminar about?

This seminar is about studying Hochschild and Cyclic Homology developed by Bökstedt, Connes and many others. Hochschild Homology was developed in 1945 while Cyclic Homology was developed by Connes in 80's. The Topological Hochschild Homology and Cyclic Homology got developed by Bökstedt, Madsen in 90's. Recently, Nikolaus and Scholze redefined these notions in the languages of ∞-categories developed by Lurie.

One of the aim of these theories is to have homology theories for associative algebras. One of the motivation for Connes was analysis non-commutative appearing in differential geometry. The other motivation is that cyclic homology(or topological cyclic homology) is best approximation of. K-theory. One of the main disadvantages of K-theory is that these groups are really hard to compute. These homology theories are explicitly defined and thus are easier to compute. One of the useful methods of computing K-groups via the cyclotomic trace map :

$$\mathrm{cyc} : K \to \mathrm{TC}.$$

The theorem of Dundas-Goodwille-McCarthy gives us explicit computations of K-groups in many cases.

The schedule of the seminar is divided into two parts:

1. Relation between cyclic homology and K-theory (7 talks).
2. Topological Cyclic Homology and its Applications (6 talks).

The first part is in relation to classical theory following from the book of Loday's *Cyclic Homology*. We shall learn the definitions of Hochschild and Cyclic Homology and its relation to K groups via Chern characters. Its is purely based on homological algebra. In fact, it is a good way to getting familiar with chain complexes and spectral sequences !!

The second part is a lot sketchier . The main aim is to get a brief idea of the functors THH and TC and try to understand its importance. We introduce classical notion of Spectra and give a brief idea of the ∞-categorical language needed to understand these functors. As a lot of participants are not familiar with this language, precise definitions for some terminologies will be skipped but the idea shall be explained of what they really mean. We shall end the seminar with two talks seeing its applications to Perfectoid rings (Bhatt-Morrow-Scholze) and Hasse-Weil Zeta function (Hesselholt).

First proposed date: 16-10-19, Wednesday, 16-18 hrs.

2 Schedule of the Seminar.

Note: * means the talks can be about 2 hours.

2.1 Relation between cyclic homology and K-theory.

14.10-20.10 Introduction to Hochschild and Cyclic Homology. Distribution of Talks.

21.10-27.10 **Hochschild Homology and trace maps.** This lecture covers the definition of Hochschild Homology and its relation to the module of relative differentials.
The main reference is [4] Chapter 1, Section 1.0-1.3.
Give a brief introduction to chain complexes. Introduce Hochschild homology via Hochschild Complex. Give elementary computation (1.1.6).Show its equivalence with Tor-definition and define normalized Hochschild complex.
Define trace maps and use it to compute Hochschild homology of matrix algebras.
Define the antisymmetrization map. State Propositions 1.3.12,1.3.15 and 1.3.16.

28.10-03.11 **Cyclic Homology I:** This lecture introduces cyclic homology and defines Connes Boundary Operator.
The main reference is [4] Chapter 2, Section 2.1-2.2.
Introduce the Cyclic bicomplex $\mathrm{HC}_\bullet(A)$. Define the Connes complex and prove theorem 2.1.5. Prove lemma 2.1.6 and explain how the Connes boundary operator B is constructed which eventually leads to the complex $\mathcal{B}(A)$. Define relative cyclic homology. Show computation of cyclic homology for any field k.
Prove and state Connes Periodicity exact sequence for unital algebras. Prove corollary 2.2.3. State and give ideas of Morita Invariance of Cyclic Homology(2.2.9) and Excision in Cyclic Homology.

4.11-10.11 **Cyclic Homology II:** This lecture gives us the connection between cyclic homology and differential forms and general notion in categorical situation.
The main reference is [4] Chapter 1-Section 1.6 ; Chapter 2-Section 2.3 and and 2.5.
Explain the relation between cyclic homology and differential forms (Section 2.3).
Introduce the notion of simplicial modules and define homology of a simplicial module (Section 1.6). Introduce the notion of cyclic modules and define $\mathcal{BC}, \bar{\mathcal{B}}\mathcal{C}$ (2.5.1-2.5.12).
If time permits, give a brief idea of mixed complexes and their properties (2.5.13-2.5.15).

11.11-17.11 **An important tool : Spectral sequences:** This lecture is devoted to a key tool used in computations of various cohomology theories: Spectral sequences.
The main reference is [11]-Chapter 5.
Motivate the definition of spectral sequences coming from a double complex (5.1).Try to explicitly show the differentials in the E_2 page for the double complex.
Define the terminologies, convergence and examples of spectral sequences with some specific conditions. Define Leray-Serre Spectral sequence (5.3) and give examples of its applications (5.3.4 ,5.3.5, 5.3.7).
Define spectral sequences associated to filtration ad state the Classical Convergence Theorem (5.4-5.5).
If time permits give the notion of hyperhomologies.

18.11-24.11 **The HKR Theorem:** The goal of this talk is to prove Hochschild-Kostant-Rosenberg (HKR) theorem.
The main reference is [4], Chapter 1-Section 3.4.
Define smooth and étale algebras and state some important properties(c.f Proposition

3.4.2).
State and prove HKR theorem. (3.4.2-3.4.9).
A main tool for computing cyclic homology for smooth and étale algebras is using Spectral Sequences. A suitable reference for introducting via bicomplex is Weibel's book.
State and prove Theorems 3.4.11 and 3.4.12.

25.11-01.12 $\mathrm{HC}^-, \mathrm{HC}^{\mathrm{per}}$ **and the Cyclic Category:*** This lecture introduces the two important variations of Cyclic homology: Negative Cyclic and Periodic Homology. It also introduces the Cyclic Category and gives a Tor intepretation of Cyclic Homology.
The main reference is [4], Chapter 5-Section 5.1 and Chapter 6-Section 6.1-6.2.
Introduce $\mathrm{HC}^-, \mathrm{HC}^{\mathrm{per}}$ and its computation for a field k. State Propositions 5.1.5 and 5.1.9. Describe the computation of these cohomologies over smooth algebras (5.1.12). Define the cyclic category ΔC and its subcategory ΔS (c.f 6.1.1-6.1.4). Describe Tor and Ext for simplicial modules (6.2.1-6.2.3). State and give a brief idea of Theorem 6.2.8.

02.12-08.12 **Generalized Chern Character:*** The aim of this talk is to define $\mathrm{ch}_n^- : H_n(\mathrm{GL}(A)) \to \mathrm{HC}_{-n}(A)$.
The main reference is [4],Chapter 8-Section 8.1-8.4.
Introduce Classical Chern Character for finitely generated projective modules (8.1.1-8.1.7). State Theorem 8.2.4 and give examples of Chern Characters(8.2.5-8.2.6).
Construct the Chern character ch_0^- (8.3.1-8.3.8) by giving brief idea of proof of theorems and example 8.3.6. State and prove Proposition 8.3.9.
Construct the Dennis Trace Map and ch_n^-(8.4.1-8.4.5). State the commutativeness of Chern characters (8.4.6). Explain the map $K_1(A) \to \mathrm{HC}_-(A)$ (8.4.7-8.4.8).

2.2 Topological Cyclic Homology and its Applications.

09.12-15.12 **An Introduction to Spectra:** This lecture introduces the classical notion of Spectra with main properties associated to it.
A preferred reference is [8] and [7].
Introduce the notion of spectra, the category of spectra. Describe the Brown Representability Theorem. Introduce K-theory,Cobordrism and Eilenberg-Maclane spaces as examples. For Eilenberg-Maclane spaces refer to [7], Page-128. Briefly give ideas of functorial properties such as long exact sequences, integration pairings and various notions of products.

16.12-22.12 **The language of ∞-categories: basic ideas and definitions:** This lecture introduces the notion of ∞-categories and basic terminologies which are analogous to classical categorical terms.
The main reference is [6], Chapter 1.
Give a brief motivation of ∞-categories by Grothendieck's hypothesis and examples such as $\mathrm{Sing}_\bullet(X)$. Define ∞-categories and Kan complexes. Define over and under-categories, initial and final objects, limits and colimits of diagrams. Define $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ where $\mathcal{C}, \mathcal{D}$ are ∞-categories. Give an idea of the term "coherent homotopy". Give a brief idea of $\mathcal{S}$- the ∞-category of spaces(c.f [6], Chapter I).

07.01.20.-12.01 **The category** Sp, E_∞**-rings and symmetric monoidal** ∞**-categories:*** This chapter introduces the ∞-category of spectra Sp, the notion of E_∞ and A_∞-rings and notion of symmetric monoidal ∞-categories.
The main reference is [5], Chapter 1-Section 1.1 and Section 1.4 ; Chapter 2.

Introduce the notion of stable ∞-categories as a classical analogue of abelian categories. Introduce the category of Sp and state some properties briefly (c.f. [5] ,Chapter 1).
State the classical definition of symmetric monoidal and monoidal categories with examples. Motivate the notion in ∞-categorical situation via "coherent homotopy". Motivate $E_\infty$and A_∞ rings by introducing classical notions of commutative algebra and associative algebra objects. Give examples of commutative ring spectrum and associative ring spectrum (for example HR for a ring R).

13.01-19.01 **The functors** THH(−) **and** TC(−)**:*** This chapter introduces THH and TC and some basic properties of this functors.
The preferred references are [1], Lecture 1, Lecture 9 and [9]-Chapter 3-5. Also the main reference [10] can be used.
Introduce THH of an associative ring spectrum. State some properties and state Bösktedt Periodicity for $\mathbb{F}_p$ (c.f [9]). State some nice properties like Corollary 4.2 of [9].
Introduce the notion of G-spectra, homotopy fixed points and homotopy orbits. Try to explain the existence of Norm map. Define the Tate Spectra. Try to explain the whole setting in the group cohomology context (c.f [1]). Introduce the cyclotomic spectra and define $\mathrm{TC}, \mathrm{TC}^-, \mathrm{TP}$ ([1],Chapter 1 ; [9], Chapter 5).
State the existence of cyclotomic trace and Dundas-Goodwille-McCarthy Theorem([1]).

20.01- 26.01 **Bökstedt Periodicity for Perfectoid Rings:*** This lecture is about computation of THH and TC for perfectoid rings and giving a brief idea of results of Bhatt-Morrow-Scholze.
The main reference is [2]-Section 1.3.
Give a brief introduction of perfectoid rings and the Fontaine period ring $\mathrm{A_{inf}}$. Describe the computation of THH of perfectoid rings resulting Bökstedt Periodicity. State the results of Bhatt-Morrow-Scholze and give an idea of the proof.

27.01-31.01 **Topological Hochschild Homology and Hasse-Weil Zeta function:*** The last lecture of the seminar concludes with an amazing paper of Hesselholt relating zeta function of smooth projective variety over $\mathbb{F}_q$ with topological periodic homology.
The reference is [3].
The outline of this talk shall be decided afterwards.

References

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[4] Jean-Louis Loday. *Cyclic homology*. Vol. 301. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Appendix E by María O. Ronco. Springer-Verlag, Berlin, 1992, pp. xviii+454. ISBN: 3-540-53339-7. DOI: 10.1007/978-3-662-21739-9. URL: https://doi.org/10.1007/978-3-662-21739-9.

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[7] J.P May. "A Concise Course in Algebraic Topology". In: (). URL: https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf.

[8] Aaron Mazel-Gee. "An Introduction to Spectra". In: (). URL: https://etale.site/writing/an-introduction-to-spectra.pdf#page132.

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[11] Charles A. Weibel. "An Introduction to Homological Algebra". In: (). URL: https://math.mit.edu/~hrm/18.906/weibel-homological-algebra.pdf#page20.