Lecture 7: Generalized Chern Character.

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Contents

1 Connections.

Assume $\mathbb{Q} \hookrightarrow K$, A commutative unital algebra (K-alg).

Definition 1.0.1. A **connection** on a A-module M is a collection of k-linear maps :

$$\nabla : M \otimes_A \Omega^n_{A/k} \to M \otimes_A \Omega^{n+1}_{A/k}$$

such that

$$\nabla(x.\omega) = (\nabla x).\omega + (-1)^n x \otimes d\omega$$

for $\omega \in \Omega^n_{A/k}$

Example 1.0.1. For example. $\nabla : M = M \otimes \Omega^0 \to M \otimes \Omega^1 \xrightarrow{\nabla} M \otimes \Omega^2$ such that $\nabla(m.a) = \nabla(m).a$.

Proposition 1.0.1. *$\nabla^2 = \nabla \circ \nabla$ is Ω^n-linear. $R := \nabla^2|_M (n = 0)$ is called the **curvature** of ∇. It is A-linear and $\nabla^2(m \otimes \omega) = R(m) \otimes \omega$.*

Example 1.0.2. If M is a A-free $R \in \text{End}_A(M) \otimes \Omega^2_{A/K}$. If M is f.g and $R \leftrightarrow M_r(\Omega^2)$.

Note: If M is f.g and proj over A. Then $R \in \text{End}_A(M) \otimes \Omega^1_{A/K}$. Morever :

$$[\nabla, -] : \text{End}_A(M) \to \text{End}_A(M) \otimes \Omega^1$$

such that $[\nabla, \alpha] = \nabla \circ \alpha - \alpha \circ \nabla$ is a connection.

Lemma 1.0.1. $[\nabla, R] = 0$.

Lemma 1.0.2. *M f.g and proj over A* $\implies$

$$\begin{array}{ccc} \mathrm{End}_A(M)\otimes\Omega^n & \xrightarrow{Q} & \mathrm{End}_A(M)\otimes\Omega^{n+1} \\ \downarrow s & & \downarrow s \\ \Omega^n & \xrightarrow{d} & \Omega^{n+1} \end{array}$$

where $Q = [\nabla, R]$ and $s = \mathrm{tr}\otimes\mathrm{id}$. commutes.

Proposition 1.0.2. *The homogeneous component of degree $2n$ of $\mathrm{tr}(\exp(R)) =: \mathrm{ch}(M,\nabla)$ is a cycle in $\Omega^{2n}_{A/K}$. Here*

$$\exp(R) = 1 + R + R^2/2! + \cdots$$

We also see that

$$[\nabla, \exp(R)] = 0$$

Theorem 1.0.1. $[\mathrm{ch}(M,\nabla)]_{dR}$ *is independent of ∇ and defined a* $\mathrm{ch}(M) \in \prod H^{2n}_{dR}(A)$.

2 Levi-Civita Connection.

We have a bijection:

$$M \text{ f.g proj of dim } l \,/\,\text{iso} \leftrightarrow e \in M_n(A) \text{ idempotent/conj by } \mathrm{GL}_l$$

One direction is given by $\mathrm{Im}\, e \leftarrow e$.
The definition of Levi-Civita connection given as follows. Let $M' \oplus m = A^l$. :

$$\nabla_e : M\otimes\Omega^n \to A^l\otimes\Omega^n_A \to A^l\otimes\Omega^{n+1}_A \to M\otimes\Omega^{n+1}_A$$

where the last map is given by $e\otimes\mathrm{id}$.

Example 2.0.1. $e \in M_l(A)$ idempotent. Then

$$\mathrm{ch}(\mathrm{Im}(e), \nabla_e) = \frac{1}{l_1}[\mathrm{tr}(e.de.de.de\cdots de)] \in \Omega^{2l}_{A/k}$$

3 Chern character for K_0

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Theorem 3.0.1.

$$\mathrm{ch}_0 : (K_0(A), \oplus, \otimes) \to (H^{ev}_{dR}(A), \oplus, \otimes)$$

defined as

$$[M] \to \mathrm{ch}(M)$$

is the Chern class of $K_0(A)$ and defines a ring homomorphism.

Example 3.0.1.

1. K_0 of vector bundles. X a topological paracompact space. We can define

   $$K^0(X) = \{\text{classes of vector bundles}\}/\sim$$

   where

   $$\sim : [E] + [E'] = [E\oplus E'].$$

   Then $(K^0(X), + = \oplus)$ is called the Grothendieck group of vector bundles over X.
   We have the following theorem:

   Theorem 3.0.2 (Serre-Swan)**.** $K^0(X) \cong K_0(\mathcal{C}_{\mathbb{C}}(X))$ *given by* $[E] \to [\Gamma(E, X)]$.

2. Chern classes for smooth varieties. $K = \mathbb{C}$. $X/\mathbb{C}$ smooth complex manifold. In this case, we have :

   $$\mathrm{ch}_0 : K^0(X) \to H^*(X, \mathbb{C}).$$

4 From classical to cyclic homology case for K_0.

We shall replace H_{dR} by H^{λ}_{2n} which is a quotient of $A^{\otimes(n+1)}/(1-t)$ where t is the cyclic operator. Eventually we shall arrive at

$$\mathrm{HC}^{-}_{2n} \xrightarrow{nat} \mathrm{HC}^{per}_{2n} \xrightarrow{nat} \mathrm{HC}_{2n}.$$

4.1 The map for H^{λ}_{2n}.

Assume: K commutative ring. 2 is regular/non-zero divisor in K.
Let $e \in M_r(A) := R$. Then $e^{\otimes(n+1)}$ maps to 0 when n is odd and $e^{\otimes n}$ when n is even.

Thus

$$e^{\otimes(n+1)} = \begin{cases} 0 & n \text{ odd} \\ \text{a cycle in } C^{\lambda}_{n}(R) \end{cases}$$

when considered as an element in $\mathrm{H}^{\lambda}_{2n}(A)$.

Theorem 4.1.1. *The map*

$$\mathrm{ch}^{\lambda}_{0,n} : K_0(A) \to H^{\lambda}_{2n}(A)$$

given by $[e] \to \mathrm{tr}((-1)^n.e^{\otimes 2n+1})$. *is well defined and functorial.*

Sketch. The map is defined as :

$$K_0(A) \to H^{\lambda}_{2n}(M(A)) \xrightarrow{\mathrm{tr}} H^{\lambda}_{2n}(A).$$

It is well defined by showing that there if $[e] = [e']$, then it maps to same element. This is shown by showing representatives of such classes in same space of matrices.

□

Claim: Upto some normalization , there exists a lift of $\mathrm{ch}^{0}_{0,n}$ to $\mathrm{HC}_{2n}(A)$.

Lemma 4.1.1. $\forall e \in M_r(A)$ *idempotent, let*

1. $y_i = (-1)^i \frac{(2i)!}{i!} e^{\otimes(2i+1)} \in M(A)^{2i+1}$.
2. $z_i := (-1)^{i-1} \frac{(2i)!}{2(i!)} (e^{\otimes(2i)}) \in M(A)^{\otimes(2i)}$.

Denote

$$c(e) := (y_n, z_n, y_{n-1}, z_{n-1}, \cdots, y_1)$$

Then $c(e) \in M(A)^{\otimes(2n+1)} \oplus \cdots \oplus M(A)$ *is a* $2n$*-cycle in* $\mathrm{Tot}(\mathrm{CC}(M(A)))$.

Theorem 4.1.2. *A unital K-algebra. Then* $\mathrm{ch}_{0,n} : K_0(A) \to \mathrm{HC}_{2n}(A)$. *Then*

$$\mathrm{ch}_{0,n}([e]) := \mathrm{tr}(c(e))$$

is well defined and functorial in A.
Morever $S \circ \mathrm{ch}_{0,n}([e]) = \mathrm{ch}_{0,n-1}([e])$ *where S is the periodicity operator.*

Corollary 4.1.1. *The chern map* $\mathrm{ch}^{\lambda}_{n}$ *and* $\mathrm{ch}_{0,n}$ *are related in the following way.*

$$p_* \mathrm{ch}_{0,n} = (-1)^n \frac{(2n)!}{n!} \mathrm{ch}^{\lambda}_{0,n}.$$

Example 4.1.1. 1. $\mathrm{ch}_{0,0} : K_0(A) \to \mathrm{HC}_0(A) = A/[A,A]$ is simply induced by the trace of the idempotent map.

2. $A = S^2_{\mathbb{C}} = \mathbb{C}[x,y,z]/(x^2+y^2+z^2-1)$. It is a classical fact : $K_0(A) = \mathbb{Z} \oplus \mathbb{Z}$ which is generated by $e := 1 + p/2 \oplus \mathbb{Z}$. Morever, we get that :

$$\mathrm{ch}_{0,2}([e]) = \mathrm{tr}(1/8pdpdp) = -i/2(xdydz + ydzdx + zdxdy)$$

which is the volume form. More generally, $\mathrm{ch}_{0,n} : S^2_{\mathbb{C}} \to H^{ev}_{dR}$ detects the fundamental generator of K_0.

Proposition 4.1.1. *There is a well defined functorial chern character map*

$$\mathrm{ch}_0^{\cdot} K_0(A) \to \mathrm{HC}_0^-(A) = \mathrm{HC}_0^{per}(A)$$

given by $\mathrm{ch}_0^-([e]) = c([e])$ *such that the following composition* ch_0 *where the composition is :*

$$K_0(A) \xrightarrow{\mathrm{ch}_0^-} \mathrm{HC}_0^-(A) \to \varinjlim \mathrm{HC}_{2n}(A) \to \cdots \to \mathrm{HC}_0(A)$$

Theorem 4.1.3. *For any k-alg A, then the composition*

$$K_0(A) \to \mathrm{HC}_0^{per}(A) \to H^{ev}_{dR}(A)$$

is upto constant the classical chern character.

Proof. Using the above proposition, the map is the Chern character corresponding to the Levi-civita connection.

□

5 Dennis trace map.

Goal $\mathrm{ch}_{n,i} : H_n(\mathrm{GL}(A)) \to \mathrm{HC}_{n+2i}(A)$ for all $n \geq 1, i \geq 0$.

Proposition 5.0.1 (Dennis trace map.). $\exists$ *a natural maps*

1.

$$\mathrm{Dt}_n : H_n(\mathrm{GL}_r(A), k) \to \mathrm{HH}_n(A)$$

which is "nice".

2. $\mathrm{ch}_n^- : H_n(\mathrm{GL}_r(A), k) \to \mathrm{HC}_n^-(A)$ *such that :*

$$\begin{array}{ccc} & & \mathrm{HC}_n^-(A) \\ & \overset{\mathrm{ch}_n^-}{\nearrow} & \downarrow h \\ H_n(\mathrm{GL}_n(A), k) & \xrightarrow{\mathrm{Dt}_n} & \mathrm{HH}_n(A) \end{array}$$

commutes.