Bridgeland Stability Conditions For Curves and Surfaces.

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SFB Meeting, February 2, 2019.

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Note: Coh(X) is an abelian subcategory of $\mathcal{D}(X)$. Are there any other abelian subcategories of $\mathcal{D}(X)$? Yes, they can be constructed by t-structures.

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2. $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for all $X \in \mathcal{D}^{\leq 0}, Y \in \mathcal{D}^{\geq 1}$.
3. For any object X in $\mathcal{D}$, there exists a distinguished triangle

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A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ is said to be *bounded* if for every object $E \in \mathcal{D}$, we have $E \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq -n}$ for $n >> 0$.

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Before moving into the stability condition, let us recall the definition of Grothendieck groups and Numerical Grothendieck groups.

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Thus $\deg(E) > 0$ which implies $Z(E) \in \mathbb{R}_{<0}$.

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Thus $\deg(E) > 0$ which implies $Z(E) \in \mathbb{R}_{<0}$.
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Let X be a smooth projective curve. We consider the standard t-structure on $\mathcal{D}(X)$. We define the function Z on $\mathcal{K}(\mathrm{Coh}(X))$ as

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It turns out that semistable objects of Z are the semistable sheaves.

Bridgeland stability conditions.

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Thus, for curves it is a complex manifold of dimesion 2 as $\mathcal{N}(X)$ is of dimension 2.

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Recall that any torsion-free sheaf E has a Harder Narasimhan filtration

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$$0 \to t(X) \to X \to X/t(X) \to 0$$
where $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$.

Recall that any torsion-free sheaf E has a Harder Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \dots E_n = E$$

where we define $\mu_i = \mu_F(E_i/E_{i-1})$ (here F is an ample divisor needed for definition of stability).

Stability conditions on surfaces.

As stated before, we need a way to construct new t-structures for the case of higher dimensions. Here X is a smooth projective surface. This is done by tilting of abelian categories. At first, we define what is a torsion pair.

Definition

Let $(\mathcal{T}, \mathcal{F})$ be a pair of full subcategories in an abelian category $\mathcal{A}$. This is said to be a **torsion pair** in $\mathcal{A}$ if the following conditions are satisfied.

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where we define $\mu_i = \mu_F(E_i/E_{i-1})$ (here F is an ample divisor needed for definition of stability). Also, we have

$$\mu_{F-max}(E) = \mu_1 > \mu_2 > \cdots > \mu_n(E) = \mu_{F-min}(E).$$

Let D, F be $\mathbb{R}$ divisors on a smooth projective surface X with F ample.

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So we have a new heart. Now we need to construct a stability function.

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That's all folks!!