BABY SEMINAR 19/20, LECTURE 1: HOCHSCHILD HOMOLOGY AND TRACE MAPS[1]

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Abstract

In this lecture we are going to define *Hochschild homology* for associative unital algebras and discuss some basic properties. Especially, we are interested in

- relation to the module of relative differentials,
- Hochschild homology of matrix algebras.

Our reference of choice will be Chapter 1 of Loday's *Cyclic Homology* ([Lod98]).

1 Presimplicial Tools

We start with a brief introduction of some homological algebra on a *presimplicial* level. Fixing this notion turns out to be useful when it comes to classical computations one encounters in homological algebra. In the following, we will simply write *module* for a module over some fixed ring, and *morphism* for a morphism in the adequate category.

Definition 1.1. A *presimplicial module* $C_.$ is a collection of modules C_n, $n \geq 0$, together with morphisms, so called *face maps* (or *face operators*),

$$d_{i,n} \colon C_n \longrightarrow C_{n-1} \quad \text{for } i \in \{0, \dots, n\}$$

such that for any n,

$$d_{i,n-1} \circ d_{j,n} = d_{j-1,n-1} \circ d_{i,n} \quad \text{for } 0 \leq i < j \leq n.$$

It may happen that we add a superscript on some face maps in order to keep track of the involved modules (if more than one).

One can turn a presimplicial module into a complex ([Lod98, 1.0.7]) by defining the boundary maps to be given by

$$d_n := \sum_{i=0}^{n} (-1)^i d_{i,n}.$$

Defining a *morphism* of presimplicial modules $f_. \colon C_. \longrightarrow C'_.$ to consist of morphisms $f_n \colon C_n \longrightarrow C'_n$ such that $f_{n-1} \circ d^C_{i,n} = d^{C'}_{i,n} \circ f_n$, one easily checks that $f_.$ can be regarded as a morphism of the complexes we have attached in the above sense.

[1]WARNING FOR ALLERGY SUFFERERS: May contain too many indices.

Definition 1.2. Let $f_., g_. \colon C_. \longrightarrow C'_.$ be morphisms of presimplicial modules. A collection $h_{.,.}$ of morphisms

$$h_{i,n} \colon C_n \longrightarrow C'_{n+1} \quad \text{for } i \in \{0, \dots n\},\ n \geq 0$$

is said to be a *presimplicial homotopy from $f_.$ to $g_.$* if for any n,

(1) $d^{C'}_{i,n+1} \circ h_{j,n} = h_{j-1,n-1} \circ d^{C}_{i,n}$ for all $i < j$,

(2) $d^{C'}_{i,n+1} \circ h_{i,n} = d^{C'}_{i,n+1} \circ h_{i-1,n}$ for all $0 < i \leq n$,

(3) $d^{C'}_{i,n+1} \circ h_{j,n} = h_{j,n-1} \circ d^{C}_{i-1,n}$ for all $i > j+1$,

(4) $d^{C}_{0,n+1} \circ h_{0,n} = f_n$,

(5) $d^{C}_{n+1,n+1} \circ h_{n,n} = g_n$.

This notion will be used when it comes to defining a homotopy between certain morphisms, because one directly computes that if $h_{.,.}$ is a presimplicial homotopy from $f_.$ to $g_.$, then the collection given by

$$h_n := \sum_{i=0}^{n} (-1)^i h_{i,n}$$

is a homotopy from $f_.$ to $g_.$.

2 Hochschild Homology

We will now introduce *Hochschild homology* an discuss some elementary facts about it. The definition we will give is the original one of Hochschild himself (instead of the definition using derived functors) since it is more closely related to the definition of *cyclic homology*, taking place in the next lectures.

Unless stated otherwise, k is a commutative ring, A is an associative unital k-algebra and M is an A-bimodule.

Definition 2.1. We define $C_n(A, M) := M \otimes A^{\otimes n}$, the tensor products taken over k, together with

$$b_n := \sum_{i=0}^{n} (-1)^i d_{i,n} \colon C_n(A, M) \longrightarrow C_{n-1}(A, M),$$

where

$$\begin{aligned} d_{0,n}(m \otimes a_1 \otimes \cdots \otimes a_n) &:= m a_1 \otimes a_2 \otimes \cdots \otimes a_n, \\ d_{i,n}(m \otimes a_1 \otimes \cdots \otimes a_n) &:= m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \quad \text{for } i \in \{1, \dots, n-1\} \\ d_{n,n}(m \otimes a_1 \otimes \cdots \otimes a_n) &:= a_n m \otimes a_2 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

As one checks that $b_n \circ b_{n+1} = 0$ by computations on the presimplicial level, we get a complex $C_.(A, M)$, called the *Hochschild complex of A with coefficients in M*. We can therefore form its homology groups, denoted $\mathrm{H}_n(A, M)$, and call them *Hochschild homology groups of A with coefficients in M*. In the case where $M = A$, we write

$$\mathrm{HH}_n(A) := \mathrm{H}_n(A, A)$$

for the *Hochschild homology groups of A*.

Remark 2.2. One may have noticed we did not use that A is unital. However, Hochschild homology for non-unital algebras is defined slightly different. We might see this later.

Taking Hochschild homology behaves functorial in the following sense. If $f\colon M \longrightarrow M'$ is an A-bimodule morphism, we get induced morphisms

$$\begin{aligned} f_n\colon \mathrm{H}_n(A,M) &\longrightarrow \mathrm{H}_n(A,M') \\ [m\otimes a_1\otimes\cdots\otimes a_n] &\longmapsto [f(m)\otimes a_1\otimes\cdots\otimes a_n]. \end{aligned}$$

On the other hand, if $g\colon A \longrightarrow A'$ is a k-algebra morphism and M' is an A'-bimodule, then we have induced morphisms

$$\begin{aligned} g_n\colon \mathrm{H}_n(A,M') &\longrightarrow \mathrm{H}_n(A',M') \\ [m\otimes a_1\otimes\cdots\otimes a_n] &\longmapsto [m\otimes g(a_1)\otimes\cdots\otimes g(a_n)], \end{aligned}$$

where M' is viewed as an A-bimodule via g. In particular we see that for $M = A$,

$$\mathrm{HH}_n\colon {}^{\text{ass,unit}}k\text{-}\mathbf{Alg} \longrightarrow k\text{-}\mathbf{Mod}$$

is a (covariant) functor. One can moreover show that

$$\mathrm{HH}_n(A\times A') = \mathrm{HH}_n(A)\oplus\mathrm{HH}_n(A').$$

Remark 2.3. One can show that the *center of* A, $Z(A) := \{z\in A : za = az \text{ for any } a\in A\}$, acts on $C_n(A,M)$ by declaring

$$z\cdot m\otimes a_1\otimes\cdots\otimes a_n := zm\otimes a_1\otimes\cdots\otimes a_n,$$

inducing a $Z(A)$-module structure on $C_n(A,M)$. One computes that this induces an endomorphism of $C.(A,M)$, in turn inducing a $Z(A)$-module structure on $\mathrm{H}_n(A,M)$. In particular, if A is commutative, $\mathrm{H}_n(A,M)$ is an A-module.

Attention. The notation of the Hochschild homology groups does not take care of the underlying ring k. However, they may change if k does (see [Lod98, 1.1.18]).

Example 2.4. Let us discuss some elementary examples.

(i) By definition one has

$$\mathrm{H}_0(A,M) = M/\{am - ma : a\in A, m\in M\},$$

which is also called the *module of coinvariants of M by A*. Therefore, denoting $[A,A]$ the commutator of A, one directly gets

$$\mathrm{HH}_0(A) = A/[A,A].$$

(ii) Under some canonical isomorphisms, the Hochschild complex of k itself is given by

$$\ldots \xrightarrow{\mathsf{id}} k \xrightarrow{0} k \xrightarrow{\mathsf{id}} k \xrightarrow{0} k.$$

Indeed, every d_i acts as the identity, and hence $b_{2n} = \mathsf{id}$ and $b_{2n+1} = 0$. Therefore,

$$\mathrm{HH}_n(k) = \begin{cases} k & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

(iii) Let $k[\epsilon] = k \oplus k\epsilon$ denote the algebra of *dual numbers* over k ($\epsilon^2 = 0$) and assume that 2 is invertible in k. Clearly, for any $n \geq 1$, $1 \otimes \epsilon^{\otimes(2n+1)}$ is a $(2n+1)$-cycle and $\epsilon \otimes \epsilon^{\otimes(2n)}$ is a $2n$-cycle. Putting some work in, one can show that the classes of those cycles span $\mathrm{HH}_{2n+1}(k[\epsilon])$, resp. $\mathrm{HH}_{2n}(k[\epsilon])$.

(iv) The next example is going to be an appetiser for trace maps and Hochschild homology of matrix algebras. We denote by $\mathcal{M}_r(A)$ the matrix algebra of $r \times r$ matrices with coefficients in A. The *abelianised* trace map $\mathcal{M}_r(A) \longrightarrow A/[A,A]$ induces an isomorphism

$$\mathcal{M}_r(A)/[\mathcal{M}_r(A), \mathcal{M}_r(A)] \cong A/[A,A],$$

so that we can use (i) to compute

$$\mathrm{HH}_0(\mathcal{M}_r(A)) \cong \mathrm{HH}_0(A).$$

Spoiler Alert.[2]

Kähler-Differentials. We now want to establish a first relation between Hochschild homology of a commutative k-algebra A and the module of relative differentials of A over k. We denote the latter by $\Omega^1_{A/k}$ - it is generated by the k-linear symbols $\mathrm{d}a$ for $a \in A$, subject to the relation

$$\mathrm{d}ab = a\,\mathrm{d}b + b\,\mathrm{d}a \quad \text{for } a, b \in A.$$

Proposition 2.5. *1. There is a canonical isomorphism* $\mathrm{HH}_1(A) \cong \Omega^1_{A/k}$.

2. If M is a symmetric A-bimodule, then $\mathrm{H}_1(A, M) \cong M \otimes_A \Omega^1_{A/k}$.

Proof. Ad 1: First of all note that, as A is commutative, $b_1 = 0$, and hence $\mathrm{HH}_1(A) = A^{\otimes 2}/\operatorname{im}(b_2)$. We define

$$\varphi\colon A^{\otimes 2} \longrightarrow \Omega^1_{A/k}, \quad a \otimes b \longmapsto a\,\mathrm{d}b.$$

Since $\varphi(b_2(a \otimes b \otimes c)) = 0$, thanks to the relation put on $\Omega^1_{A/k}$, we obtain

$$\varphi\colon\ \mathrm{HH}_1(A) \longrightarrow \Omega^1_{A/k}.$$

Conversely, defining $\psi(a\,\mathrm{d}b) := [a \otimes b] \in \mathrm{HH}_1(A)$ gives rise to a map $\Omega^1_{A/k} \longrightarrow \mathrm{HH}_1(A)$, since

$$\begin{aligned} \psi(a\,\mathrm{d}b + b\,\mathrm{d}a) - \psi(\mathrm{d}ab) &= [a \otimes b - 1 \otimes ab + b \otimes a] \\ &= [b_2(1 \otimes a \otimes b)]. \end{aligned}$$

Clearly, φ and ψ are mutually inverse.

Ad 2: Now, as M is symmetric, $b_1 = 0$, and hence $\mathrm{H}_1(A, M) = M \otimes A/\operatorname{im}(b_2)$. We define

$$\varphi\colon M \otimes A \longrightarrow M \otimes_A \Omega^1_{A/k}, \quad m \otimes a \longmapsto m \otimes \mathrm{d}a.$$

Since $\varphi(b_2(m \otimes a_1 \otimes a_2)) = 0$, thanks to symmetry of M and, again, the relation put on $\Omega^1_{A/k}$, we obtain

$$\varphi\colon\ \mathrm{H}_1(A, M) \longrightarrow M \otimes_A \Omega^1_{A/k}.$$

On the other hand, one can verify similarly to the above that defining

$$\psi\colon M \otimes_A \Omega^1_{A/k} \longrightarrow \mathrm{H}_1(A, M), \quad m \otimes a\,\mathrm{d}b \longmapsto [ma \otimes b]$$

makes sense (respecting the relation put on $\Omega^1_{A/k}$). Clearly these maps are mutually inverse. □

[2] We will see the resulting relation between the 0-th homology groups in (iv) a little later for the homology groups in higher degree when discussing the so called *generalised* trace map.

2.1 **Equivalence with Tor-definition.** Now we are going to compare the definition of Hochschild homology given above and the definition in terms of the (derived) Tor-functors. To do so, we note that for any n, $A^{\otimes n}$ has a (left) A^{e}-module structure given by

$$(\lambda \otimes \lambda') \cdot a_1 \otimes \cdots \otimes a_n := \lambda a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n \lambda'$$

The comparison is done by the so-called *bar resolution*, which is constructed as follows.

Definition 2.6. We define $C_n^{\mathrm{bar}}(A) := C_{n+1}(A) = A^{\otimes(n+2)}$, together with

$$b'_n := \sum_{i=0}^{n} (-1)^i d_{i,n+1} \colon C_n^{\mathrm{bar}}(A) \longrightarrow C_{n-1}^{\mathrm{bar}}(A),$$

with the maps $d_{i,n}$ taken from the definition of the Hochschild complex.

As one checks that $b'_n \circ b'_{n+1} = 0$, the above definition gives a complex $C_\cdot^{\mathrm{bar}}(A)$ of $A^{\mathrm{e}}-$*modules*, called the *bar complex of* A.

Proposition 2.7. *The complex $C_\cdot^{\mathrm{bar}}(A)$ gives a resolution of the A^{e}-module A:*

$$C_\cdot^{\mathrm{bar}}(A) \xrightarrow{\mu} A \longrightarrow 0,$$

where $\mu\colon A \otimes A \longrightarrow A$ is the multiplication map.

Proof. We have to show that the homology groups of the augmented complex vanish. Since A is unital, μ is surjective, and it is also clear that $\ker(\mu) = \mathrm{im}(b'_1)$. Our aim is now to define a contracting homotopy for $C_\cdot^{\mathrm{bar}}(A)$. As the vanishing of the homology groups has nothing to do with the module structure we put on it, we consider $C_\cdot^{\mathrm{bar}}(A)$ for a moment as complex of A^{op}-modules under $A^{\mathrm{op}} \longrightarrow A \otimes A^{\mathrm{op}}$, sending $\lambda \mapsto 1 \otimes \lambda$. We claim that

$$\begin{aligned} s_n \colon C_n^{\mathrm{bar}}(A) &\longrightarrow C_{n+1}^{\mathrm{bar}}(A) \\ a_0 \otimes \cdots \otimes a_{n+1} &\longmapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n+1} \end{aligned}$$

is the saught for contracting homotopy. We start calculating

$$\begin{aligned} b'_{n+1} \circ s_n + s_{n-1} \circ b'_n &= \sum_{i=0}^{n+1} (-1)^i d_{i,n+2} \circ s_n + \sum_{j=0}^{n} (-1)^j s_{n-1} \circ d_{i,n+1} \\ &= d_{0,n+2} \circ s_n + \sum_{i=0}^{n} (-1)^{i+1} d_{i+1,n+2} \circ s_n - \sum_{j=0}^{n} (-1)^{j+1} s_{n-1} \circ d_{i,n+1}. \end{aligned}$$

As clearly $d_{0,n+2} \circ s_n = \mathsf{id}_{A^{\otimes(n+2)}}$, computing $d_{i+1,n+2} \circ s_n = s_{n-1} \circ d_{i,n+1}$ finishes the proof. □

Remark 2.8. In the previous proof we used that A has a left unit. It A has a right unit, we should take a slightly different s_n (putting the 1 on the right).

Proposition 2.9. *If A is projective as a k-module, then there is an isomorphism (of groups)*

$$\mathrm{H}_n(A, M) \cong \mathrm{Tor}_n^{A^{\mathrm{e}}}(M, A).$$

Proof. Since the k-projectivity of A implies the k-projectivity of $A^{\otimes n}$, it follows from a general fact that $C_n^{\mathrm{bar}}(A) = A \otimes A^{\otimes n} \otimes A$ is A^{e}-projective. Indeed, let M be any k-module and N be an A^{e}-module. Then it is easily seen that the defining an isomorphism

$$\begin{aligned} \mathrm{Hom}_{A^{\mathrm{e}}}(A \otimes M \otimes A, N) &\longrightarrow \mathrm{Hom}_k(M, N) \\ g &\longmapsto (m \mapsto g(1 \otimes m \otimes 1)) \end{aligned}$$

induces a natural isomorphism

$$\mathrm{Hom}_{A^{\mathrm{e}}}(A \otimes M \otimes A, -) \xRightarrow{\sim} \mathrm{Hom}_k(A, -) \circ \mathsf{F},$$

where F is a forgetful functor. Now, putting $M = A^{\otimes n}$: As $\mathrm{Hom}_k(A^{\otimes n}, -)$ is exact by assumption (and F clearly is, too), this gives that $A \otimes A^{\otimes n} \otimes A$ is A^{e}-projective. Therefore, the augmented complex

$$C^{\mathrm{bar}}_{\cdot}(A) \longrightarrow A \longrightarrow 0$$

is a projective resolution of A as an A^{e}-module. Applying $M \otimes_{A^{\mathrm{e}}} -$ yields

$$\ldots \longrightarrow M \otimes_{A^{\mathrm{e}}} A^{\otimes(n+2)} \xrightarrow{\mathsf{id} \otimes b'_n} M \otimes_{A^{\mathrm{e}}} A^{\otimes(n+1)} \xrightarrow{\mathsf{id} \otimes b'_{n-1}} \ldots \longrightarrow M \otimes_{A^{\mathrm{e}}} A \longrightarrow 0,$$

which is used to compute $\mathrm{Tor}_n^{A^{\mathrm{e}}}(M, A)$. Under the isomorphism given by

$$\begin{aligned} M \otimes_{A^{\mathrm{e}}} A^{\otimes(n+2)} &\longrightarrow M \otimes A^{\otimes n} \\ m \otimes a_0 \otimes \cdots \otimes a_{n+1} &\longmapsto a_0 m a_{n+1} \otimes a_1 \otimes \cdots \otimes a_n, \end{aligned}$$

one computes that the maps $\mathsf{id} \otimes b'_n$ correspond to b_n, finishing the proof. □

Normalised Hochschild Complex. The Hochschild complex contains a large subcomplex which is acyclic and at some point it might be helpful to get rid of it. To do so, let D_n be the submodule of $C_n(A, M)$ generated by the so-called *degenerate elements*:

$$m \otimes a_1 \otimes \cdots \otimes a_n \quad \text{with } a_i = 1 \text{ for some } i.$$

Definition 2.10. We define $\bar{C}_{\cdot}(A, M) := C_n(A, M)/D_n$ and call the resulting complex (together with the induced boundary maps) *normalised Hochschild complex.*

One can show that $D_{\cdot}$ forms an acyclic complex and that $C_{\cdot}(A, M)$ is quasi-isomorphic to $\bar{C}_{\cdot}(A, M)$ (see [Lod98, 1.1.15])

2.2 **Trace Maps and Morita Invariance.** Now we want to extend the trace map for matrices to the Hochschild complex. As we will see, this *generalised trace map* induces an isomorphism on the level of homology. This is established by a theorem called *Morita Invariance for Matrices*, allowing us to compute Hochschild homology of matrix algebras.

Definition 2.11. The *n-th generalised trace map* is defined to be

$$\begin{aligned} \mathrm{tr}_n \colon C_n(\mathcal{M}_r(A), \mathcal{M}_r(M)) &\longrightarrow C_n(A, M) \\ \alpha^{(0)} \otimes \alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)} &\longmapsto \sum_{(\ell_0, \ldots, \ell_n)} \alpha^{(0)}_{\ell_0, \ell_1} \otimes \alpha^{(1)}_{\ell_1, \ell_2} \otimes \cdots \otimes \alpha^{(n)}_{\ell_n, \ell_0}, \end{aligned}$$

where the sum runs through the set $\{1, \ldots, r\}^{n+1}$.

Identifying $\mathcal{M}_r(M)$ (resp. $\mathcal{M}_r(A)$) with $\mathcal{M}_r(k) \otimes M$ (resp. $M_r(k) \otimes A$), every element of $\mathcal{M}_r(M)$ (resp. $\mathcal{M}_r(A)$) is a sum of elements of the form ua with $u \in \mathcal{M}_r(k)$ and $a \in M$ (resp. $a \in A$).

Lemma 2.12 ([Lod98, 1.2.2]). *Let $u_i \in \mathcal{M}_r(k)$, $a_0 \in M$, $a_j \in A$, $i \geq 0$, $j \geq 1$. Then*

$$\mathrm{tr}(u_0 a_0 \otimes \cdots \otimes u_n a_n) = \mathrm{tr}(u_0 \cdots u_n) a_0 \otimes \cdots \otimes a_n.$$

This helps us verifying the following

PROPOSITION 2.13. *The generalised trace maps induce a morphism of the associated Hochschild complexes*

$$\mathrm{C}.(\mathcal{M}_r(A), \mathcal{M}_r(M)) \longrightarrow \mathrm{C}.(A, M).$$

Proof. As mentioned earlier, we are reduced to check things on the presimplicial level, meaning that for any n,

$$d_{i,n} \circ \mathrm{tr}_n = \mathrm{tr}_{n-1} \circ d_{i,n}^{\mathcal{M}} \quad \text{for any } i.$$

This in turn is to be checked only on elements of the form $u_0a_0 \otimes \cdots \otimes u_na_n$ as in the above lemma. We compute

$$\begin{aligned} d_{i,n}(\mathrm{tr}_n(u_0a_0 \otimes \cdots \otimes a_n)) &= \mathrm{tr}(u_0 \cdots u_n) d_{i,n}(a_0 \otimes \cdots \otimes a_n) \\ &= \begin{cases} \mathrm{tr}(u_0 \cdots u_n) a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & \text{if } 0 \leq i \leq n-1 \\ \mathrm{tr}(u_0 \cdots u_n) a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{if } i = n, \end{cases} \end{aligned}$$

while on the other hand

$$\begin{aligned} \mathrm{tr}_{n-1}(d_{i,n}^{\mathcal{M}}(u_0a_0 \otimes \cdots \otimes u_na_n)) &= \begin{cases} \mathrm{tr}_{n-1}(u_0a_0 \otimes \cdots \otimes u_ia_iu_{i+1}a_{i+1} \otimes \cdots \otimes u_na_n) & \text{if } 0 \leq i \leq n-1 \\ \mathrm{tr}_{n-1}(u_na_nu_0a_0 \otimes u_1a_1 \otimes \cdots \otimes u_{n-1}a_{n-1}) & \text{if } i = n \end{cases} \\ &= \begin{cases} \mathrm{tr}(u_0 \cdots u_n) a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes a_n & \text{if } 0 \leq i \leq n-1 \\ \mathrm{tr}(u_n u_0 \cdots u_{n-1}) a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{if } i = n, \end{cases} \end{aligned}$$

finishing the proof. □

The next theorem builds a bridge between the Hochschild homologies of the matrix algebras and the Hochschild homologies of the underlying algebra. For this we introduce the *n-th inclusion map* to be given by

$$\begin{aligned} \mathrm{inc}_n \colon C_n(A, M) &\longrightarrow C_n(\mathcal{M}_r(A), \mathcal{M}_r(M)) \\ m \otimes a_1 \otimes \cdots \otimes a_n &\longmapsto E_{1,1}(m) \otimes E_{1,1}(a_1) \otimes \cdots \otimes E_{1,1}(a_n). \end{aligned}$$

It is easily seen that this induces a morphism of the corresponding Hochschild complexes.

THEOREM 2.14 (Morita Invariance[3] for Matrices). *The induced morphisms on Hochschild homology*

$$\mathrm{tr}_{*,n} \colon \mathrm{H}_n(\mathcal{M}_r(A), \mathcal{M}_r(M)) \longrightarrow \mathrm{H}_n(A, M)$$

and

$$\mathrm{inc}_{*,n} \colon \mathrm{H}_n(A, M) \longrightarrow \mathrm{H}_n(\mathcal{M}_r(A), \mathcal{M}_r(M))$$

are mutually inverse isomorphisms.

Proof. Clearly $\mathrm{tr}. \circ \mathrm{inc}. = \mathsf{id}_{C.(A,M)}$ and hence

$$\mathrm{tr}_{*,n} \circ \mathrm{inc}_{*,n} = (\mathrm{tr}. \circ \mathrm{inc}.)_{*,n} = \mathsf{id}_{\mathrm{H}_n(A,M)}.$$

To complete the proof, it is now sufficient to construct a homotopy

$$\mathrm{inc}. \circ \mathrm{tr}. \sim \mathsf{id}_{C.(\mathcal{M}_r(A), \mathcal{M}_r(M))}.$$

[3]Why *Morita Invariance*? There is a general framework for a notion called *Morita Equivalence* (named after the Japanese mathematician Kiiti Morita), constituting a relationship between rings that preserves many ring-theoretic properties. But we will not get into this at this point.

This in turn will be done by constructing a presimplicial homotopy $h_{\cdot,\cdot}$ between the maps of interest. Given n, for any $i \in \{0, \dots, n\}$ we define

$$h_{i,n} \colon C_n(\mathcal{M}_r(A), \mathcal{M}_r(M)) \longrightarrow C_n(\mathcal{M}_r(A), \mathcal{M}_r(M))$$

by sending $\alpha^{(0)} \otimes \cdots \otimes \alpha^{(n)}$ to

$$\sum_{(\ell_0,\dots,\ell_{i+1})} E_{\ell_0,1}(\alpha^{(0)}_{\ell_0,\ell_1}) \otimes E_{1,1}(\alpha^{(1)}_{\ell_1,\ell_2}) \otimes \cdots \otimes E_{1,1}(\alpha^{(i)}_{\ell_i,\ell_{i+1}}) \otimes E_{1,\ell_{i+1}}(1) \otimes \alpha^{(i+1)} \otimes \cdots \otimes \alpha^{(n)},$$

where the sum runs through the set $\{1, \dots, r\}^{i+2}$. Now there are some calculations to do. We will not present every computation here. Just to illustrate what is going on, we will verify (1) for $i = 0$, $j = 1$, and (4) and (5). On the one hand,

$$\begin{aligned} d^{\mathcal{M}}_{0,n+1}(h_{1,n}(\alpha^{(0)} \otimes \cdots \otimes \alpha^{(n)})) &= d^{\mathcal{M}}_{0,n+1}\Big(\sum_{(\ell_0,\ell_1,\ell_2)} E_{\ell_0,\ell_1}(\alpha^{(0)}_{\ell_0,\ell_1}) \otimes E_{1,1}(\alpha^{(1)}_{\ell_1,\ell_2}) \otimes E_{1,\ell_2}(1) \otimes \alpha^{(2)} \otimes \cdots \otimes \alpha^{(n)}\Big) \\ &= \sum_{(\ell_0,\ell_1,\ell_2)} E_{\ell_0,1}(\alpha^{(0)}_{\ell_0,\ell_1}\alpha^{(1)}_{\ell_1,\ell_2}) \otimes E_{1,\ell_2}(1) \otimes \alpha^{(2)} \otimes \cdots \otimes \alpha^{(n)}, \end{aligned}$$

while on the other hand,

$$\begin{aligned} h_{0,n-1}(d^{\mathcal{M}}_{0,n}(\alpha^{(0)} \otimes \cdots \otimes \alpha^{(n)})) &= \sum_{(\ell_0,\ell_1)} E_{\ell_0,1}((\alpha^{(0)}\alpha^{(1)})_{\ell_0,\ell_1}) \otimes E_{1,\ell_1}(1) \otimes \alpha^{(2)} \otimes \cdots \otimes \alpha^{(n)} \\ &= \sum_{(\ell_0,\ell_1)} \sum_{\ell} E_{\ell_0,1}(\alpha^{(0)}_{\ell_0,\ell}\alpha^{(1)}_{\ell,\ell_1}) \otimes E_{1,\ell_1}(1) \otimes \alpha^{(2)} \otimes \cdots \otimes \alpha^{(n)}. \end{aligned}$$

Ad (4):

$$\begin{aligned} d^{\mathcal{M}}_{0,n+1}(h_{0,n}(\alpha^{(0)} \otimes \cdots \otimes \alpha^{(n)})) &= \sum_{(\ell_0,\ell_1)} d^{\mathcal{M}}_{0,n+1}(E_{\ell_0,1}(\alpha^{(0)}_{\ell_0,\ell_1}) \otimes E_{1,\ell_1}(1) \otimes \alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)}) \\ &= \sum_{(\ell_0,\ell_1)} E_{\ell_0,1}(\alpha^{(0)}_{\ell_0,\ell_1})E_{1,\ell_1}(1) \otimes \alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)} \\ &= \sum_{(\ell_0,\ell_1)} E_{\ell_0,\ell_1}(\alpha^{(0)}_{\ell_0,\ell_1}) \otimes \alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)} \\ &= \alpha^{(0)} \otimes \alpha^{(1)} \otimes \cdots \otimes \alpha^{(n)}. \end{aligned}$$

Ad (5):

$$\begin{aligned} &d^{\mathcal{M}}_{n+1,n+1}(h_{n,n}(\alpha^{(0)} \otimes \cdots \otimes \alpha^{(n)})) \\ &= \sum_{(\ell_0,\dots,\ell_{n+1})} d^{\mathcal{M}}_{n+1,n+1}(E_{\ell_0,1}(\alpha^{(0)}_{\ell_0,\ell_1}) \otimes E_{1,1}(\alpha^{(1)}_{\ell_1,\ell_2}) \otimes \cdots \otimes E_{1,1}(\alpha^{(n)}_{\ell_n,\ell_{n+1}}) \otimes E_{1,\ell_{n+1}}(1)) \\ &= \sum_{(\ell_0,\dots,\ell_{n+1})} E_{1,\ell_{n+1}}(1)E_{\ell_0,1}(\alpha^{(0)}_{\ell_0,\ell_1}) \otimes E_{1,1}(\alpha^{(1)}_{\ell_1,\ell_2}) \otimes \cdots \otimes E_{1,1}(\alpha^{(n)}_{\ell_n,\ell_{n+1}}) \\ &= \sum_{\substack{(\ell_0,\dots,\ell_{n+1}): \\ \ell_0=\ell_{n+1}}} E_{1,1}(\alpha^{(0)}_{\ell_0,\ell_1}) \otimes E_{1,1}(\alpha^{(1)}_{\ell_1,\ell_2}) \otimes \cdots \otimes E_{1,1}(\alpha^{(n)}_{\ell_n,\ell_{n+1}}) \\ &= \sum_{(\ell_0,\dots,\ell_n)} E_{1,1}(\alpha^{(0)}_{\ell_0,\ell_1}) \otimes E_{1,1}(\alpha^{(1)}_{\ell_1,\ell_2}) \otimes \cdots \otimes E_{1,1}(\alpha^{(n)}_{\ell_n,\ell_0}) \\ &= \mathrm{inc}_n\Big(\sum_{(\ell_0,\dots,\ell_n)} \alpha^{(0)}_{\ell_0,\ell_1} \otimes \alpha^{(1)}_{\ell_1,\ell_2} \otimes \cdots \otimes \alpha^{(n)}_{\ell_n,\ell_0}\Big) \\ &= \mathrm{inc}_n(\mathrm{tr}_n(\alpha^{(0)} \otimes \cdots \otimes \alpha^{(n)})). \end{aligned}$$

This finishes what we wanted to show. □

Taking $M = A$, this gives an isomorphism of the Hochschild homologies

$$\mathrm{HH}_n(\mathcal{M}_r(A)) \cong \mathrm{HH}_n(A).$$

Remark 2.15. One can show that HH_n commutes with inductive limits, so that even

$$\mathrm{HH}_n(\mathcal{M}_\infty(A)) \cong \mathrm{HH}_n(A).$$

Here, $\mathcal{M}_\infty(A) := \mathsf{lim}_r\, \mathcal{M}_r(A)$, where we embed $\mathcal{M}_r(A) \hookrightarrow \mathcal{M}_{r+1}(A)$ by bordering the matrix with zeros from the right and below.

2.3 **The Antisymmetrisation Map.** Now we want to introduce a map that might show up again when studying the relation between differential forms and *cyclic* homology.

Given $a_0 \otimes \cdots \otimes a_n \in C_n(A, M)$ and $\sigma \in S_n$, we define

$$\sigma \cdot a_0 \otimes \cdots \otimes a_n := a_0 \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$$

and extend this action by k-linearity to the algebra $k[S_n]$.

Definition 2.16. We define the *antisymmetrisation element* to be

$$\varepsilon_n := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma \in k[S_n].$$

The induced map, which we still denote ε_n, given by

$$\begin{aligned} \varepsilon_n \colon M \otimes \bigwedge^n A &\longrightarrow C_n(A, M) \\ a_0 \otimes a_1 \wedge \cdots \wedge a_n &\longmapsto \varepsilon_n \cdot a_0 \otimes \cdots \otimes a_n, \end{aligned}$$

is called the *antisymmetrisation map*.

Remark 2.17. Of course, one should observe that the above is well-defined by showing that if $a_i = a_j$ for some $1 \le i < j \le n$, then

$$\varepsilon_n \cdot a_0 \otimes \cdots \otimes a_n = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_0 \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)} = 0.$$

The antisymmetrisation maps are related to the Hochschild complex by the following commutative diagram ([Lod98, 1.3.5])

$$\begin{array}{ccc} M \otimes \bigwedge^n A & \xrightarrow{\varepsilon_n} & C_n(A, M) \\ \big\downarrow{\scriptstyle \delta_n} & & \big\downarrow{\scriptstyle b_n} \\ M \otimes \bigwedge^{n-1} A & \xrightarrow{\varepsilon_{n-1}} & C_{n-1}(A, M). \end{array} \tag{2.1}$$

Here, δ_n is the *Chevalley-Eilenberg map*, defined by sending $a_0 \otimes \cdots \otimes a_n \in C_n(A, M)$ to

$$\sum_{i=1}^{n} (-1)^i [a_0, a_i] \otimes a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_n$$
$$+ \sum_{1 \le i < j \le n} (-1)^i a_0 \otimes [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge \widehat{a_j} \wedge \cdots \wedge a_n,$$

where $[a, b] := ab - ba$. In particular, if A is commutative and M is symmetric, then $b_n \circ \varepsilon_n = 0$.

Coming to an end, we state two propositions, in which we assume A to be commutative.

Proposition 2.18 ([Lod98, 1.3.12 and 1.3.15]). *1. The antisymmetrisation map induces a canonical map, still denoted by the same symbol,*

$$\varepsilon_n \colon M \otimes_A \Omega^n_{A/k} \longrightarrow \mathrm{H}_n(A, M).$$

2. The canonical map

$$C_n(A, M) \longrightarrow M \otimes_A \Omega^n_{A/k}$$
$$a_0 \otimes \cdots \otimes a_n \longmapsto a_0 \otimes \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n$$

induces a map

$$\pi_n \colon \mathrm{H}_n(A, M) \longrightarrow M \otimes_A \Omega^n_{A/k}.$$

In particular, for $M = A$, we have induced maps $\varepsilon_n \colon \Omega^n_{A/k} \longrightarrow \mathrm{HH}_n(A)$ and $\pi_n \colon \mathrm{HH}_n(A) \longrightarrow \Omega^n_{A/k}$.

As one might guess, the last statement we are going to mention describes the relation between the induced maps π_n and ε_n.

Proposition 2.19 ([Lod98, 1.3.16]). *The composition $\pi_n \circ \varepsilon_n$ acts as multiplication by $n!$ on $M \otimes_A \Omega^n_{A/k}$. Therefore, if k contains $\mathbb{Q}$, then $M \otimes_A \Omega^n_{A/k}$ is a direct summand of $\mathrm{H}_n(A, M)$.*

References

[Lod98] Jean-Louis Loday. *Cyclic homology.* Springer-Verlag Berlin Heidelberg GmbH, 1998.