Lecture 8: Introduction to Spectra.

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1 Stable Homotopy theory.

(all spaces are pointed).

Definition 1.0.1. For a topological space (X, x_0),its (reduced)**suspension** ΣX is defined as

$$\Sigma X = SX/\{x_0\} \times I$$

Here

$$SX = \frac{X \times I}{X \times \{0, .1\}}$$

Remark 1.0.1. One can see that $\Sigma X = S^1 \wedge X = S^1 \times X/S^1 \vee X$. So we see that. $\Sigma S^n = S^1 \wedge S^n = S^{n+1}$ for all $n \geq 0$.

Definition 1.0.2. The **loopspace** ΩX of a space X is the spaces of based loops in X

Lemma 1.0.1. *There is an adjunction*

$$[\Sigma X, Y] = [X, \Omega Y]$$

where $[-,-]$ *means the homotopy class of maps. Roughly, the map is given by*

$$(f : \Sigma X \to Y) \mapsto (x \to f|_{\{x\}\times[0,1]})$$

Note that the suspension Σ gives rise to maps

$$\sigma : \pi_n X \to \pi_{n+1}\Sigma X$$

given by

$$[f : S^n \to X] \to [\Sigma f : S^{n+1} \to \Sigma X]$$

This is a well defined homomorphism.

Theorem 1.0.1. *(Freuderthal) Let* X *be a pointed* CW *complex Fix a positive integer* n*. Then* $\forall i \geq n+2$, $\pi_{n+i}(\Sigma^i X)$ *are all isomorphic.*

Definition 1.0.3. The stable limit in the above theorem is called the **nth stable homotopy group** of X and is denoted by $\pi_n^s(X)$, i.e

$$\pi_n^s(X) = \varinjlim_i \pi_{n+i}(\Sigma^i X).$$

Remark 1.0.2. By Serre, $\pi_n^s(S^0)$ is finite for all $n \geq 0$. But in general it is very hard to compute.

2 The Category of Spectra.

Definition 2.0.1. A CW spectrum (simply spectrum) is a sequence $\{E_n\}_{n\in\mathbb{Z}}$ of CW complexes together with structure maps $\Sigma E_n \to E_{n+1}$ (which maybe. inclusions as subcomplexes).

Example 2.0.1.

1. The suspension spectrum $\Sigma^\infty X$ of a CW complex X is defined by

$$(\Sigma^\infty X)_n = \Sigma^n X$$

2. The sphere spectrum $\mathbb{S}$ is the suspension spectrum of S^0.
3. An Ω spectrum is a sequence of CW complexes E_n with a weak homotopy equivalencs $E_n \to \Omega E_{n+1}$. By adjunction, every Ω spectrum defines a spectrum.
4. (Eilenberg-Maclane spaces) Fix an integer $n > 0$ and a group F. An Eilenberg-Maclane space is a space $K(G, n)$ with

$$\pi_i(K(G,n)) = \begin{cases} G & i = n \\ 0 & \text{otherwise} \end{cases}$$

The most important property of Eilenberg Maclane-spaces is that they "represent" cohomology in the sense that :

$$H^n(X;G) \cong [X, K(G,n)]$$

So we see that, $\pi_n(\Omega(K(G,n+1)) = [S^n, OmegK(G,n+1)] = \pi_{n+1}(K(G,n+1)) = G$. Hence we must have a weak homotopy equivalence. This turns out that the collection of Eilenberg-Maclane spaces for a given group G into a spectrum HG where $\text{HG}_n = K(G,n)$.

Definition 2.0.2. A function f of degree r between spectra E and F is a collection of maps $f_n : E_n \to F_{n-r}$ commuting with structure maps :

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\Sigma f_n} & \Sigma F_{n-r} \\ \downarrow & & \downarrow \\ E_{n+1} & \xrightarrow{f_{n+1}} & F_{n-r+1} \end{array}$$

Definition 2.0.3. A subspectrum $E' \subset E$ is **cofinal**. when every cell in E_m is eventually mapped to a cell in some E'_{m+N}.

Definition 2.0.4. Let E ,F be spectra and U, V be two cofinal subspectra of E. Let $f : U \to F$ and $g : V \to f$ be two functors of spectra. Then f and g are equivalent if they. agree on the cofinal subspectrum $U \cap V$. Then f and g are equivalent if they agree on the cofinal subspectrum $U \cap V$. A **map** from E to F is an equivalence class of such functions.

Definition 2.0.5. Let I^+ be the unit interval with disjoint basepoint added. Let E be a spectrum. The cylinder spectrum $\mathrm{Cyl}(E)$ of E has $(\mathrm{Cyl}(E))_n = i^+ \wedge E_n$ and structure maps given by

$$\Sigma(I^+ \wedge E)_n = I^+ \wedge \Sigma E_n \to I^+ \wedge E_{n+1}$$

Definition 2.0.6. We say two maps $f, g : E \to F$ are **homotopic** if there is a map $\mathrm{Cyl}(E) \to F$ restricting to f and g at the ends of the cylinder. Homotopy equivalence is an equivalence relation and a morphism of spectra is a homotopy class of maps.

Definition 2.0.7. With this definition, the collection of spectra becomes a homotopy category, called the **"stable homotopy category"** and it is denoted by Spe.

Remark 2.0.1. 1. $(\mathrm{Spe}, \wedge)$ is a symmetric monoidal category and its unit object is the sphere spectrum $\mathbb{S}$ i.e $E \wedge \mathbb{S} = E$.

Definition 2.0.8. The homotopy groups of a spectrum E is defined to be $\pi_n(E) := [\Sigma^n \mathbb{S}, E]_0 = [\mathbb{S}, E]_n$.

Proposition 2.0.1. *If E is a spectrum, then $\pi_n(E) = \varinjlim_k \pi_{n+k}(E_k)$.*

Corollary 2.0.1. $\pi_n(\Sigma^\infty X) = \pi_n^s(X)$.

3 Cohomology.

If E is a spectrum and X a CW complex, then $X \wedge E$ is the spectrum with $(X \wedge E)_n = X \wedge E_n$ with obvious structure maps.

Definition 3.0.1. Let E be a spectrum. Define the E-cohomology of a CW complex X to be $E^n X := [\Sigma^\infty X, E]_{-n}$.

Theorem 3.0.1 (Brown Representability.). *Every generalised cohomology theory on a connected* CW *complex is an E-cohomology for some spectrum E.*

Example 3.0.1. 1. The Ω spectrum HA gives $\mathrm{HA}^n x = [\Sigma^\infty X, HA]_{-n} = [\Sigma^{-n}\Sigma^\infty x, \mathrm{HA}]_0 = [X, \mathrm{K}(A, n)] = H^n(X, A)$.

2. The complex K-theory. We know that $K^0(X)$ is represented by $BU \times \mathbb{Z}$ where BU:classifying space of the unitary group U. This means $K^0(X) = [X, \mathrm{BU} \times \mathbb{Z}]$. in this way, we see that $K^{-1}(X) = [X, \Omega(BU \times \mathbb{Z})]$.

   Theorem 3.0.2 (Bott Complex Periodicity). $\Omega^2(BU \times \mathbb{Z}) = \mathrm{BU} \times \mathbb{Z}$.

   We get that complex K-theory is 2-periodic.