Comparison Theorem - Crystalline vs. de Rham

Ran Azouri
23/12/2020

Bhatt - De Jong

Notations and Reminder

$\Sigma = \mathrm{Spec}(\mathbb{Z}_p)$, S - a Σ-scheme with p-locally nilpotent at S

$(S/\Sigma)_{cris}$ - The small crystalline site

objects: (U, T, δ) — U: open UCS; T: UCT nilp. thickening; δ: divided power structure on I_{UT}

$\mathcal{O}_{S/\Sigma}$

$\mathcal{O}_{S/\Sigma}((U,T,\delta)) := \Gamma(T, \mathcal{O}_T)$

$\Omega^1_{S/\Sigma}$

For B a $\mathbb{Z}_p/p^e$-algebra, and an ideal $J \subset B$ with div. powers δ,

$\Omega^1_B :=$ Module of Σ-linear differentials $/ d\delta_n(x) = \delta_{n+1}(x)\,dx$ (compatibility with divided powers), $x \in J$

$\Omega^1_{S/\Sigma}((U,T,\delta)) := \Gamma(T, \Omega^1_T)$

An $\mathcal{O}_{S/\Sigma}$-module $\mathcal{F}$ on $(S/\Sigma)_{cris}$ is <u>quasi-coh</u> if $\forall (U,T,\delta)$, the restriction $\mathcal{F}_T$ is q-coh $\mathcal{O}_T$-module.

$\mathcal{F}$ is a <u>crystal</u> in q-coh modules if in addition:

$\forall f : (U,T,\delta] \longrightarrow (U',T',\delta')$

$c_f : f^* \mathcal{F}_{T'} \to \mathcal{F}_T$ is an iso.

Ex: $\mathcal{O}_{S/\Sigma}$ is a crystal, $\Omega^i_{S/\Sigma}$ is q-coh but not a crystal

For every crystal we have a canonical connection

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega^1_{S/\Sigma}$$

It is integrable, meaning that we have a complex

$$\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}} \Omega^1_{S/\Sigma} \to \mathcal{F} \otimes \Omega^2_{S/\Sigma} \to \cdots$$

De Rham complex

Let A be a $\mathbb{Z}/p^N$ algebra, $S = \mathrm{Spec}(A)$ [Affine case]

choose a polynomial alg $/\mathbb{Z}_p$, P, & $0 \to J \to P \twoheadrightarrow A \to 0$

J: the kernel

Define $D = D_J(P)^{\wedge}$

$D_J(P)$: Divided power envelope; $\wedge$: p-adic completion

Fact: Equivalence of categories

$$\left\{ \begin{array}{c} \text{crystals in qcoh} \\ \mathcal{O}_{S/\Sigma}\text{-modules} \end{array} \right\} \simeq \{ (M, \nabla) \}$$

$\simeq$: eq. of categories; M: D-module (D the ring above); ∇: A topologically qu-nilp integrable connection $M \to M \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}}$

$$F \longmapsto (F(0), \nabla)$$

$$F(0) = \lim_{e \geq N} F((S, \mathrm{Spec}(D/p^e D), \delta)) = \lim_{e \geq N} F(X_e)$$

Čech-Alexander complex

$$F(\cdot) := \left(F(0) \to F(1) \to F(2) \to \cdots \right)$$

$$\lim_e F(X_e) \quad \lim_e F(X_e \times_{(S/\Sigma)_{cris}} X_e) \quad \lim_e F(X_e \times X_e \times X_e) \quad \cdots$$

More explicit description: $0 \to J \to P \twoheadrightarrow A \to 0$ as above.

Set $J(n) = \ker(\underbrace{P \otimes \cdots \otimes P}_{(n+1)\text{ times}} \to A)$

$$D(n) = D_{J(n)}(P \otimes \cdots \otimes P)^{\wedge}$$

$$(S, \mathrm{Spec}(D(n)/p^e), \delta(n)) = \underbrace{X_e \times \cdots \times X_e}_{(n+1)\text{-times}}$$

($\times$: $(S/\Sigma)_{cris}$)

So the n-th term in the complex is

$$F(n) = \lim_{e \geq N} F((S, \mathrm{Spec}(D(n)/p^e), \delta(n)))$$

Fact: Let F be a qcoh $\mathcal{O}_{S/\Sigma}$-module, and assume:

⊛ $\forall n>0 \quad R^1 \lim_{e\to\Delta} F(X_e^{\times(n+1)}) = 0$, then the complex $F(\bullet)$ computes $R\Gamma(S/\Sigma; F)$

⊛ is satisfied for q-coh crystals & for $\Omega^i_{S/\Sigma}$, as $F(X_e^{\times(n+1)}) \longrightarrow F(X_{e-1}^{\times(n-1)})$ is surjective then.

§ **Main theorem (affine case):**

Let F a crystal in qcoh $\mathcal{O}_{S/\Sigma}$-modules corr. to (M, ∇).

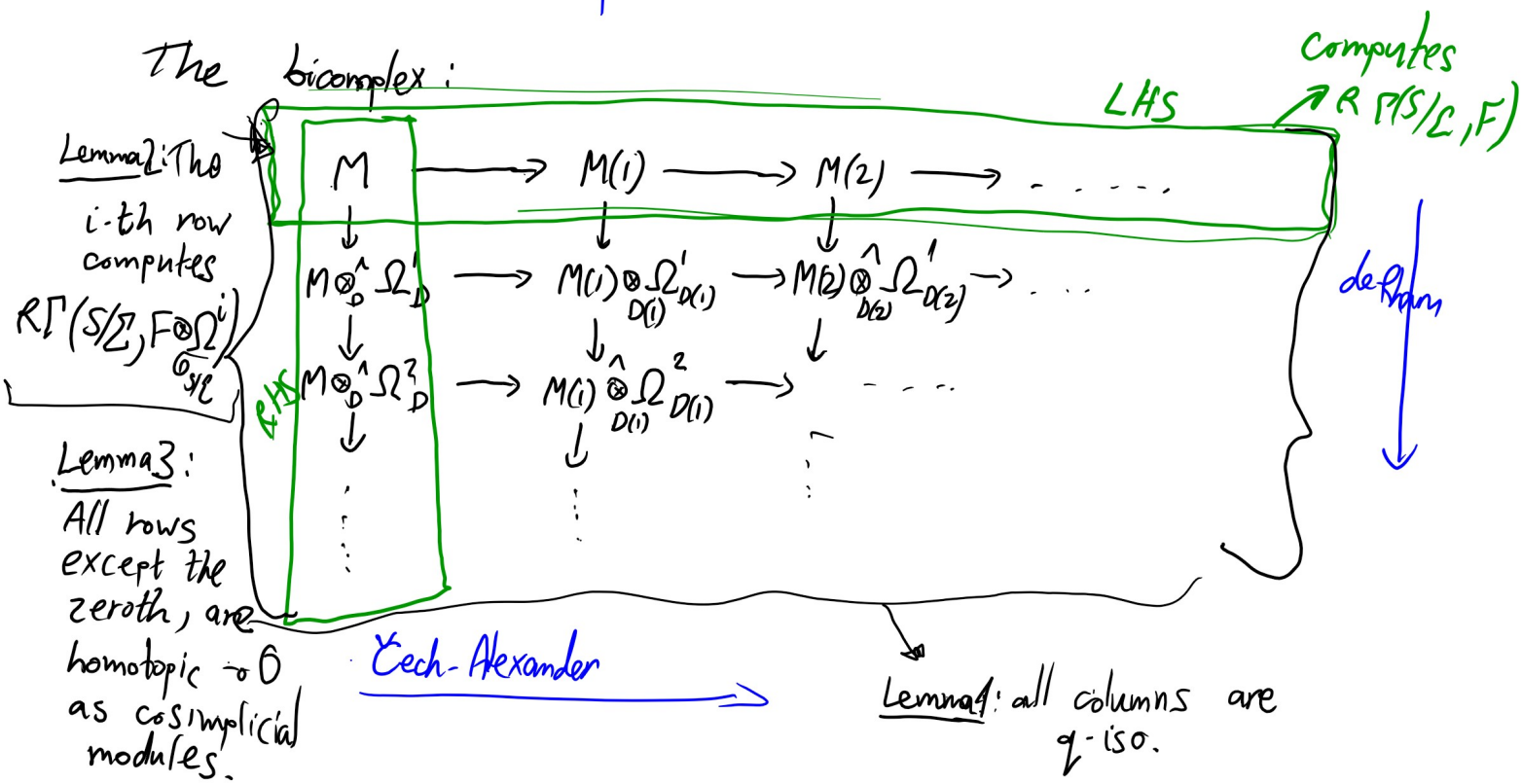
Then

$$R\Gamma(S/\Sigma, F) \simeq \left(M \to M\hat{\otimes}_D \Omega^1_D \to M\hat{\otimes}_D \Omega^2_D \to \dots\right)$$

crystalline coh, can be computed by Čech-Alexander complex

de-Rham complex

The bicomplex:



From the lemmas we conclude –

$$\nearrow H^p_{vert}\left(H^q_{hor}(C_{\cdot\cdot})\right) \Rightarrow H^\cdot(Tot(C_{\cdot\cdot}))$$
$$\searrow H^p_{hor}\left(H^q_{ver}(C_{\cdot\cdot})\right)$$

$C_{p,q}$

$C_{\cdot\cdot}$ → $C_{\cdot,0}$, $C_{0,\cdot}$

From lemma 1 → all isomorphic

$$H^m(C^{0,\cdot}) \xrightarrow{0} H^m(C^{1,\cdot}) \xrightarrow{1} H^m(C^{2,\cdot}) \xrightarrow{0}$$

The other spectral seq gives the crystalline coh $R\Gamma(S/\Sigma, F)$ (lemma 3) LHS.

→ This gives de-Rham cohom → complex RHS □

Proof of lemma 1: Claim that $M \hat{\otimes}_D \Omega^\cdot_D \longrightarrow M(n) \hat{\otimes}_{D(n)} \Omega^\cdot_{D(n)}$ induced by $D \underset{D''(0)}{\longrightarrow} D(n)$ is a q.iso.

idea: There's a filt of $M(n) \hat{\otimes}_{D(n)} \Omega^\cdot_{D(n)}$ w. graded pieces $M \otimes_D \Omega^i_D \otimes_D \Omega^\cdot_{D(n)/D}$

it reduces to showing that

$$D \longrightarrow (D(n) \to \Omega^1_{D(n)/D} \to \Omega^2_{D(n)/D} \to \dots) \text{ is q.is.}$$

can be checked directly.

· Pf of lemma 2: The i-th row computes $R\Gamma(S/\Sigma, F \otimes \Omega^i_{S/\Sigma})$

$$M \otimes \Omega^i_D \to M(1) \otimes_{D(1)} \Omega^i_{D(1)} \to \dots$$

Alg fact: $\left\{ \begin{matrix} p\text{-adically complete} \\ \mathbb{Z}_p\text{-modules} \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{proj. systems} \\ (M_e, \phi_e) \end{matrix} \right\}$

$\mathbb{Z}/p^e$-module $\phi_e : M_{e+1}/p^e \xrightarrow{\sim} M_e$

$$M \hat{\otimes}_D \Omega^i_D \simeq \lim_{e \geq N} \left(M/p^e \otimes_{D/p^e} \Omega^i_{D/p^e} \right) \simeq$$

$$\simeq \lim_{e \geq N} \left((F \otimes \Omega^\cdot_{S/\Sigma})(S, \mathrm{Spec}(D/p^e), \delta) \right)$$

similarly with each $D(n)$

The fact that $M/p^{e+1} \otimes_{D/p^{e+1}} \Omega^i_{D/p^{e+1}} \to M/p^e \otimes_{D/p^e} \Omega^i_{D/p^e}$ surj.

$\otimes$ is satisfied $\Rightarrow$ This computes $R\Gamma(\mathcal{X}, F \otimes \Omega^i)$.

Pf of lemma 3: All rows >0 are htpic to 0.

i-th row $M \otimes_D \Omega^i_D \to M(1) \otimes_{D(1)} \Omega^i_{D(1)} \to \cdots$

Enough to show that

$$\Omega^1_D \to \Omega^1_{D(1)} \to \Omega^1_{D(2)} \to \cdots \quad \text{is htpic to } 0$$

since Λ^i, $M(\cdot) \otimes$) ↖ p-adic comp

will leave it still htpic to 0.

Enough to check

$M_\bullet = (\Omega'_P \to \Omega'_{P \otimes P} \to \cdots)$ is htpic to 0.

since our complex is base-changing $M_\bullet$ with $P^{\otimes(n+1)} \to D(n)$

$$P = \mathbb{Z}_p[\{X_i\}_{i \in I}]$$

$P^{\otimes(n+1)}$ = polynomials on $1 \otimes \cdots \otimes \underset{\uparrow}{X_i} \otimes 1 \otimes \cdots \otimes 1 =: X_i(e)$

e-th place

$\Omega'_{P^{\otimes(n+1)}}$ is free on gen. $dX_i(e)$

cosimp str looks like: $f: [n] \to [m]$ $M_\bullet(f)(dX_i(e)) = dX_i(f(e))$

$$M_n = \bigoplus_{i \in I} \left(\bigoplus_{e: 0 \to n} P^{\otimes(n+1)} dX_i(e) \right)$$

M_n: $i \in I$; $dX_i(f(e))$

Key htpical alg Lemma:

Suppose $A_\bullet$ cosimp ring

Define a cosimp. module $M_\bullet$ by $M_n = \bigoplus_{e: 0 \to n} A_n ke$

$f: [n] \to [m]$ $M_\bullet(f): M_n \to M_m$

$k_e \longmapsto k_{f(e)}$

then $M_\bullet$ is htpk to $O_\bullet$

one can construct explicitely $h : M_\bullet \longrightarrow \mathrm{Hom}(\Delta[1], M_\bullet)$ s.t $ev_0 \circ h = 0$, $ev_1 \circ h = id_M$ //

More facts:

Fact 1: $\left\{ \begin{array}{c} \text{crystals on} \\ \text{qc oh modules} / O_{S/\Sigma} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{crystals of} \\ \text{q-coh mod} / \mathcal{O}_S \otimes_{\mathrm{Spec}(\mathbb{F}_p)} / \Sigma \end{array} \right\}$

To prove that: comparing M^i_S in (M, ∇) $\quad S = \mathrm{Spec} A$

compare div. power envelopes of $P/p^e \longrightarrow A$

$\wr$

div. power envelopes $P/p^e \longrightarrow A \longrightarrow A/p$

Fact 2: We have a morphism of topoi

$u_{S/\Sigma} : \left(Sh(S/\Sigma)_{cris}, O_{S/\Sigma} \right) \longrightarrow \left(Sh(S_{Zar}), f^{-1} O_\Sigma \right)$

$f : S \to \Sigma$

$$u^{-1}_{S/\Sigma}(F)(U, T, \delta)) = F(U)$$

$\hookrightarrow$ gives $Ru_{S/\Sigma} : D(O_{S/\Sigma}) \longrightarrow D(f^{-1} O_\Sigma)$

s.th for all $U \in S_{Zar}$ $R\Gamma(U, Ru_{S/\Sigma}(F)) = R\Gamma(U/\Sigma, F)$

The fact itself: S/Σ p is loc. nilp on S, F crystal

$$\left(F \to F \otimes_{O_{S/\Sigma}} \Omega^1_{S/\Sigma} \to F \otimes \Omega^2_{S/\Sigma} \to \dots \right) \to F[0]$$

is a q.iso, after applying $Ru_{S/\Sigma}$.

we have to show $Ru_{S/\Sigma}\left(F \otimes \Omega^i_{S/\Sigma}\right) = 0$ $\leftarrow$ comes from lemma 2 in the proof.

Global setting: Let S be a Σ-scheme s.t. $p^N = 0$ on S. Assume $S \hookrightarrow X$ (fin. presentation & sm $/\Sigma$)

D_e : div. power envelope for $\mathcal{O}_X/p^e \longrightarrow \mathcal{O}_S$

T - formal scheme with $\mathcal{O}_T = \varinjlim_{e \in \mathbb{N}} D_e$ $\quad T_e = \mathrm{Spec}(D_e)$

can similarly define (M, ∇) $\quad$ F is a crystal in $\mathcal{O}_{S/\Sigma}$-modules

Thm: The hypercoh of $M \to M \otimes_{\mathcal{O}_T} \Omega^1_T \to M \otimes_{\mathcal{O}_T} \Omega^2_T \to \cdots$ computes $R\Gamma(S/\Sigma, F)$

Cor: Let $f: X \to \Sigma$ be a proper sm. morphism of schemes, $S = X \times_{\Sigma} \mathrm{Spec}(\mathbb{F}_p)$, then

$\mathcal{O}_X \to \Omega^1_{X/\Sigma} \to \Omega^2_{X/\Sigma} \to \cdots$ computes $R\Gamma(S/\Sigma, \mathcal{O}_{S/\Sigma})$