Lecture 2: Cyclic Homology I.

Antonio Mejiás Gil.

30th October 2019.

1 Aim of the talk

1. Construction of cyclic homology.
2. Rel to Hochschild hom and further results.

2 Bicomplexes.

A associative algebra over k a commutative ring.

Definition 2.0.1. A bicomplex is a collection of A-modules $C_{pq}, p, q \in \mathbb{Z}$ together with

$$d^h : C_{pq} \to C_{p-1q}$$

and

$$d^v : C_{pq} \to C_{p(q-1)}$$

such that

1. $d^h d^h = d^v d^v = 0$
2. $d^h d^v + d^v d^h = 0$

Remark 2.0.1. This is almost the same as having complex of complexes, but it has anti-commutativity.

Definition 2.0.2. The **total complex** of a bicomplex $\mathrm{CC}_\bullet$ is

$$\mathrm{Tot}(CC)_n = \bigoplus_{p+q=n} C_{pq}$$

with boundary morphisms:

$$d : \mathrm{Tot}(CC)_n \to \mathrm{Tot}(CC)_{n-1}$$

given by

$$d(\sum_{p+q=n} a_{pq})_{p'q'} = d^h a_{(p'+1)q'} + d^v a_{p'(q'+1)}$$

Note: we are considering only non-negative bicomplexes.
This can be proved to be a complex.

3 Cyclic bicomplex.

Recall that from last talk we had $C_n(A = A^{\otimes(n+1)}$ are the Hochschild Homology groups.

This group has a "cyclic action" by $\mathbb{Z}/(n+1)\mathbb{Z}$. Let t be the generator the group.
We denote

$$t.(a_0 \otimes ... a_n) = (-1)^n (a_n \otimes a_0 \otimes ... a_{n-1}).$$

Definition 3.0.1. The **norm operator** is given by $N = 1 + t + t^2 + \cdots + t^n$.

Lemma 3.0.1. *1.* $(1-t)b' = b(1-t)$.

2. $b'N = Nb$.

Definition 3.0.2. The cyclic bicmplex $\mathrm{CC}_{\bullet\bullet}(A)$ with

$$\mathrm{CC}_{pq} = C_q = A^{(q+1)}.$$

and the homomorphisms are defined as :

1. $d^h_{pq} = \begin{cases} N, & p \text{ even} \\ (1-t), & `p \text{ odd} \end{cases}$

2. $d^v_{pq} = \begin{cases} b, & p \text{ even} \\ -b' & p \text{ odd} \end{cases}$

It looks like:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
A^{\otimes 3} & \xleftarrow[1-t]{} & A^{\otimes 3} & \xleftarrow[N]{} & A^{\otimes 3} & & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow[1-t]{} & A^{\otimes 2} & \xleftarrow[N]{} & A^{\otimes 2} & & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A & \xleftarrow[1-t]{} & A & \xleftarrow[N]{} & A & & \cdots
\end{array}$$

Note this is a bicomplex by the previous lemma.

Definition 3.0.3. The **cyclic homology groups** are

$$\mathrm{HC}_n(A) := H_n(\mathrm{Tot}(CC_{\bullet\bullet}(A))$$

Remark 3.0.1. Functoriality on A i.e $f : A \to A'$ induces a morphism $f_* : \mathrm{HC}_\bullet(A) \to \mathrm{HC}_\bullet(A')$.

We are gonna see the different construction of CycliC Homology groups.
Recall, we have a Hochschild complex

$$\cdots C_n(A) \xrightarrow{b} C_{n-1}(A) \cdots$$

Induces a map :

$$\cdots C^\lambda_n(A) := C_n(A)/(1-t) \xrightarrow{b} C^\lambda_{n-1}(A) \cdots.$$

This is by the previous lemma.

Definition 3.0.4. This complex $C^{\lambda}_{\bullet}(A)$ is called the **Connes complex**. We denote $H^{\lambda}_n(A) = H_n(C^{\lambda}_{\bullet}(A))$.

We define :

$$p_n : \mathrm{Tot}(\mathrm{CC}(A))_n \to C^{\lambda}_n(A)$$

defined by

$$\sum_{p+q=n} a_{pq} \to a_{0n} + (1-t)C_n(A)$$

Theorem 3.0.1. *If $k \supset \mathbb{Q}$, then p_n is a quasi-isomorphism.(using $1/(n+1) \subset \mathbb{Q}$)*

Lemma 3.0.2 (Killing Contractible complexes.). *Let $\cdots A_n \oplus A'_n \xrightarrow{d} A_{n-1} \oplus A'_{n-1} \to \cdots$ be a complex given by*

$$d = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

such that (A_*, δ) is contractible with a contracting homotopy h i.e its satisfies $h\delta + \delta h = \mathrm{id}$. Then

$$i = (\mathrm{id}, -h\gamma) : (A_*, \alpha - \beta h\gamma) \to (A_* \oplus A'_*, d)$$

is a quasi-isomorphism.

Proof. First, let us see that i is a homomorphism of complexes.

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} . \begin{bmatrix} \mathrm{id} \\ -h\gamma \end{bmatrix} = \begin{bmatrix} \alpha - \beta h\gamma \\ \gamma - \delta h\gamma \end{bmatrix}$$

and

$$\begin{bmatrix} \mathrm{id} \\ -h\gamma \end{bmatrix} . \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha - \beta h\gamma \\ -h\gamma\alpha + h\gamma\beta h\gamma \end{bmatrix}$$

We have $d^2 = 0$. This implies,we have :

1. $\alpha^2 \beta\gamma = 0$
2. $\alpha\beta + \beta\delta = 0$
3. $\gamma\alpha + \delta\gamma = 0$
4. $\gamma\beta = 0$.

Thus :

$$\begin{aligned} &\gamma - \delta h\gamma \\ &= (h\delta + \delta h)\gamma - \delta h\gamma \\ &= \delta h\gamma \\ &= -h\gamma\alpha \end{aligned}$$

Also notice that $\gamma\beta = 0$ and implies the equality and proves it is a morphism of complexes.
Now the coker of the mpa is given by $(A_{\bullet}, \delta)$. But (A_*, δ) is contractible, i.e it i has trivial homomlogy in positive dimension. Thus i is a quasi-isomorphism.

□

4 Constructing $\mathcal{B}$.

Suppose A is unital :

1. Consider the column $p = 1$ in $CC(A)$ i.e with $-b'$. Denote this by $\mathcal{C}$
2. One can view the total complex as
$$\mathrm{Tot}(\mathrm{CC}(A)) \cong \mathcal{K} \oplus \mathcal{C}$$
3. Apply the previous lemma to $d = \begin{bmatrix} b & 1-t \\ n & -b' \end{bmatrix}$, since $\mathcal{C}$ is contractible, with homotopy s (extra degeneracy map), we get a quasi-isomorphism $\mathcal{K} \hookrightarrow \mathrm{Tot}(\mathrm{CC}(A))$.
4. We obtain $B : \mathrm{CC}_{2q} \to \mathrm{CC}_{0(q+1)}$.
5. We obtain a new bicomplex by iterating the process:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & \\
A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A & & \\
\downarrow b & & \downarrow b & & & & \\
A^{\otimes 2} & \xleftarrow{B} & A & & & & \\
\downarrow b & & & & & & \\
A & & & & & &
\end{array}$$

Definition 4.0.1. Denote $B = (1-t)sN$ to be the **Connes Boundary Operator.**

Theorem 4.0.1. *A unitial, then :*

$$B : \mathrm{Tot}(\mathcal{B}(A)) \hookrightarrow \mathrm{Tot}(\mathrm{CC}(A))$$

is a quasi-isomorphism.

5 Rel to Hochschild Homology and further results.

Theorem 5.0.1. *A arbitrary. There exists a long exact sequence:*

$$\cdots \mathrm{HH}_n(A) \xrightarrow{I} \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \xrightarrow{B} \mathrm{HH}_{n-1}(A) \cdots$$

This comes from

$$0 \to C_\bullet(A) \xrightarrow{I} \mathrm{Tot}(\mathcal{B}(A)) \to \mathrm{Tot}(\mathcal{B}(A))[-2] \to 0$$

where I is given by identification of $C_\bullet(A)$ with the first column of $\mathcal{B}(A)$. The cokernel is the complex shifted by -2, this follows from the diagram.

Corollary 5.0.1. *If $f : A \to A'$ induces an isom in Hochschild homology , then it induces an iso in cyclic homology.*

The idea comes from writing long exact sequences of A and A' and using induction and five lemma.

6 Morita invariance and Excision.

We have generalized trace map :

$$\mathrm{tr} : \mathcal{M}_r(A)^{\otimes n} \to A^{\otimes n}$$

We can see that the action of t is compttible with this map.

Theorem 6.0.1. *The map :*

$$\mathrm{tr}_\bullet : \mathrm{HC}_\bullet(\mathcal{M}(A)) \to \mathrm{HC}_\bullet(A)$$

is an isomorphism.

Theorem 6.0.2. *Let* $0 \to I \to A \to A/I \to 0$ *be an exact sequence of* k*-algebra such that* A *and* A/I *are unitial. Then there exists a long exact sequence:*

$$\cdots \mathrm{HC}_n(I) \to \mathrm{HC}_n(A) \to \mathrm{HC}_n(A/I) \to \mathrm{HC}_{n-1}(I) \to \cdots$$

The idea of this proof is to introduce relative cyclic homology groups $\mathrm{HC}_n(I) \cong \mathrm{HC}_n(A.I)$.

Theorem 6.0.3. *Let* A *be unital and commutative. Then*

$$\mathrm{HC}_1(A) \cong \Omega^1_{A/k}/dA$$

When $A = k$, the vanishing of hochschild homology for k implies the periodicity for k.