Divided Powers: II

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Outline

P.D. envelope

We start with the "P.D. envelope" of an ideal, which is the P.D. analogues of formlal completion.

Theorem (Existence & Universal property of P.D. envelope)

Given

- (A, I, γ): *a P.D. ring.*
- B: *an* A*-algebra (via* $f : A \to B$*).*
- $J \subset B$: *an ideal.*

There exists a B*-P.D. algebra* $(\mathcal{D}_{B,\gamma}(J), \bar{J}, [\ ])$ *(via* $\psi : B \to \mathcal{D}_{B,\gamma}(J)$*) such that*

- $J\mathcal{D}_{B,\gamma}(J) \subset \bar{J}$*.*
- $[\ ]$ *compatible with* γ *(via* $\psi \circ f$*).*

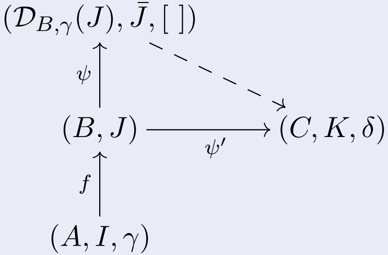
satisfies the following universal property:

Theorem (Existence & Universal property of P.D. envelope)

For any B*-P.D. algebra* (C, K, δ) *(via* $\psi' : B \to C$*) satisfying*

- $JC \subset K$.
- δ *compatible with* γ *(via* $\psi' \circ f$*).*

There exists a unique P.D. morphism $(\mathcal{D}_{B,\gamma}(J), \bar{J}, [\ ]) \to (C, K, \delta)$ *making the following diagram commute*



This B-P.D. algebra $(\mathcal{D}_{B,\gamma}(J), \bar{J}, [\ ])$ is called *the P.D. envelope of* J *in* B *relative to* (A, I, γ).

Proof: Special case

Existence

Assume $IB \subset J$. In this case, both $\psi \circ f$ and $\psi' \circ f$ are P.D. morphism. The natural candidate for $(\mathcal{D}_{B,\gamma}(J), \bar{J}, [\ ])$ is the Roby's algebra

$$(\Gamma_B(J), \Gamma_B^+(J), [\ ]) \text{ plus } \psi : J \to \Gamma_B^1(J) \cong J \subset \Gamma_B(J)$$

But there may be two problems:

1. Is $J\Gamma_B(J) \subset \Gamma_B^+(J)$, i.e.

$$\text{Is } \underbrace{\psi(x)}_{\deg=1} - \underbrace{x}_{\deg=0} = 0 \text{ for } x \in J?$$

   Let $\mathscr{J}_1 \subset \Gamma_B(J)$ be the ideal generated by elements of this form.

2. Is $[\ ]$ compatible with γ (via $\psi \circ f$), i.e.

$$\text{Is } \psi(f(y))^{[n]} - \psi(f(\gamma_n(y))) = 0 \text{ for any } y \in I?$$

   Let $\mathscr{J}_2 \subset \Gamma_B(J)$ be the ideal generated by elements of this form.

Proof: Special case

Existence *cont.*

Let $\mathscr{I} := \mathscr{I}_1 + \mathscr{I}_2$ and define

$$(\mathcal{D}_{B,\gamma}(J), \bar{J}, [\ ]) := (\Gamma_B(J)/\mathscr{I}, \Gamma_B^+(J)/\mathscr{I}, [\ ])$$

This will solve these two problems. BUT! It brings a new problem:

Is $(\Gamma_B(J)/\mathscr{I}, \Gamma_B^+(J)/\mathscr{I}, [\ ])$ a B-P.D. algebra?

Note: this is a quotient ring of a P.D. ring. By Lemma (P.D. structure on a quotient ring), need to check

$\mathscr{I} \cap \Gamma_B^+(J) \subset \Gamma_B^+(J)$ is a sub-P.D. ideal

Since $\mathscr{I}_2 \subset \Gamma_B^+(J)$, we have $\mathscr{I} \cap \Gamma_B^+(J) = \mathscr{I}_1 \cap \Gamma_B^+(J) + \mathscr{I}_2$. Using the formula $\gamma_n(x+y)$, it suffices to show

$x^{[n]} \in \mathscr{I} \cap \Gamma_B^+(J)$ if $x \in \mathscr{I}_1 \cap \Gamma_B^+(J)$ or $x \in \mathscr{I}_2$

Proof: Special case

Existence *cont.*

If $x \in \mathscr{I}_1 \cap \Gamma_B^+(J)$, say

$$x = \sum a_i(\underbrace{\psi(x_i)}_{\deg=1} - \underbrace{x_i}_{\deg=0}) \text{ with } a_i \in \Gamma_B(J)$$

Since $x \in \Gamma_B^+(J)$, the degree 0 part of this sum vanishes. Write $a_i = a_i^0 + a_i^+$ with $a_i^0 \in \Gamma_B^0(J) = B$ and $a_i^+ \in \Gamma_B^+(J)$, then

$$\sum a_i^0 x_i = 0 \text{ and hence } 0 = \psi\left(\sum a_i^0 x_i\right) = \sum a_i^0 \psi(x_i)$$

i.e. $\sum a_i^0(\psi(x_i) - x_i) = 0$. Then

$$x = \sum a_i^+(\psi(x_i) - x_i) \in \mathscr{I}_1 \Gamma_B^+(J)$$

This shows $\mathscr{I}_1 \cap \Gamma_B^+(J) = \mathscr{I}_1 \Gamma_B^+(J)$ and it is easily seen to be a sub-P.D. ideal of $\Gamma_B^+(J)$. Then $x^{[n]} \in \mathscr{I}_1 \cap \Gamma_B^+(J) \subset \mathscr{I} \cap \Gamma_B^+(J)$.

Proof: Special case

Existence *cont.*

If $x \in \mathscr{J}_2$, by Lemma (being a sub-P.D. ideal can be checked on a generating set), it suffices to see

$$\Big(\psi(f(y))^{[n]} - \psi(f(\gamma_n(y)))\Big)^{[m]} \in \mathscr{J} \cap \Gamma_B^+(J) \text{ for } y \in I \text{ and } m \geq 1$$

Clearly, it is in $\Gamma_B^+(J)$. It's remaining to show it is in $\mathscr{J}$ (to simplify, write $x^{[n]}$ for $\psi(x)^{[n]}$): LONG computation involved!

Proof: Special case

Existence *cont.*

$$
\begin{aligned}
\left(f(y)^{[n]} - f(\gamma_n(y))^{[1]}\right)^{[m]} &= \sum_{r+s=m} (-1)^s (f(y)^{[n]})^{[r]} (f(\gamma_n(y))^{[1]})^{[s]} \text{ (A2)} \\
&= \sum_{r+s=m} (-1)^s C_{r,n} f(y)^{[nr]} f(\gamma_n(y))^{[s]} \text{ (A5)} \\
&\equiv \sum_{r+s=m} (-1)^s C_{r,n} f(\gamma_{nr}(y)) f(\gamma_s(y)\gamma_n(y)) \\
&\equiv f\left(\sum_{r+s=m} (-1)^s \gamma_r(\gamma_n(y))\gamma_s(\gamma_n(y))\right) \text{ (A5)} \\
&\equiv f\left((\gamma_n(y) - \gamma_n(y))^{[m]}\right) \text{ (A2)} \\
&\equiv 0 \pmod{\mathscr{I}}
\end{aligned}
$$

Proof: Special case

Universal property

Assume $IC \subset K$. In this case, by the universal property of $(\Gamma_B(J), \Gamma_B^+(J), [\ ])$, there is a P.D. morphism

$$(\Gamma_B(J), \Gamma_B^+(J), [\ ]) \to (C, K, \delta)$$

Then check it factors through $(\Gamma_B(J)/\mathscr{J}, \Gamma_B^+(J)/\mathscr{J}, [\ ])$.

Proof: General case

- (Existence) Apply Special case to $f : (A, I, \gamma) \to (B, J + IB)$, we obtain a B-P.D. algebra

$$(\mathcal{D}_{B,\gamma}(J + IB), \overline{J + IB}, [\ ])$$

and let $\bar{J} \subset \overline{J + IB}$ be the sub-P.D. ideal generated by J. Then

$$(\mathcal{D}_{B,\gamma}(J + IB), \bar{J}, [\ ])$$

is what we want.

- (Universal property) Apply Special case to

$$(A, I, \gamma) \xrightarrow{f} (B, J + IB) \xrightarrow{\psi'} (C, K + IC, \delta')$$

we obtain a P.D. morphism

$$(\mathcal{D}_{B,\gamma}(J + IB), \bar{J}, [\ ]) \to (C, K + IC, \delta')$$

but $\bar{J}C$ (the sub-P.D. ideal generated by JC) is already in K.

P.D. envelope

Representability reformulation

Theorem (Existence & Universal property of P.D. envelope)

Given

- (A, I, γ)*: a P.D. ring.*
- B*: an* A*-algebra (via* $f : A \to B$*).*
- $J \subset B$*: an ideal.*

Then the functor $F : \mathcal{PD} \to \mathcal{S}et$ *defined by*

$$(C, K, \delta) \mapsto \left\{ \psi' \middle| \begin{array}{c} \psi' : (B, J) \to (C, K) \text{ an } A\text{-alg morp with } JC \subset K \text{ s.t.} \\ \psi' \circ f : (A, I, \gamma) \to (C, K, \delta) \text{ is a morp of P.D. rings} \end{array} \right\}$$

is representable, where $\mathcal{PD}$ *is the category of* B*-P.D. algebras.*

One by-product

Lemma (Criteria for extension of P.D. structure)

The followings are equivariant:

- *A P.D. structure on IB compatible with γ.*
- *A section of the canonical map $B \to \mathcal{D}_{B,\gamma}(IB)$ such that if K is its kernel, then $K \cap \overline{IB} \subset \overline{IB}$ is a sub-P.D. ideal.*

Proof.

Exercise for universal property. □

One example

If M is an A-mod, $B := \mathrm{Sym}_A^{\bullet}(M)$, $J := \mathrm{Sym}_A^{+}(M)$, then

$$\mathcal{D}_{B,0}(J) = \Gamma_A(M)$$

where 0 means the trivial P.D. structure on $(0) \subset A$.
In particular, if M is a free A-mod with basis $\{x_1, \ldots, x_n\}$, then $B = A[x_1, \ldots, x_n]$ and $J = (x_1, \ldots, x_n)$, then

$$\mathcal{D}_{B,0}(J) = A\langle x_1, \ldots, x_n \rangle$$

is the P.D. polynomial algebra.

Important remarks

(1) $\mathcal{D}_{B,\gamma}(J)$ only depends on $J + IB$, while $\bar{J}$ still depends on J.
☐ By the very construction of $\mathcal{D}_{B,\gamma}(J)$.

(2) If the structure map $A \to B$ factors through some A' and γ extends to γ' on A', then

$$\mathcal{D}_{B,\gamma}(J) = \mathcal{D}_{B,\gamma'}(J)$$

☐ By the very construction of $\mathcal{D}_{B,\gamma}(J)$, the base ring A only play a role in $\mathscr{I}_2$, where there is obviously no difference between choosing those y from I or IA'.

(3) As a B-algebra, $\mathcal{D}_{B,\gamma}(J)$ is generated by $\{x^{[n]} : n \geq 0, x \in J\}$,
☐ This was already true for $\Gamma_B(J)$.
Any set of generators of J gives a set of P.D. generators of $\bar{J}$.
☐ This is trivial.

Important remarks

(4) The canonical map $B/J \to \mathcal{D}_{B,\gamma}(J)/\bar{J}$ is an isomorphism iff γ extends to B/J, e.g. if I is principal or $IB \subset J$.
☐ $\Rightarrow$ Trivial. $\Leftarrow$ By the universal property of $(\mathcal{D}_{B,\gamma}(J), \bar{J}, [\ ])$.

(5) If γ extends to B/J and $B \to B/J$ has a section, then $\mathcal{D}_{B,\gamma}(J) = \mathcal{D}_{B,0}(J)$, i.e. we can drop the compatibility condition.
☐ The idea is to construct an inverse to the canonical surjective map $\mathcal{D}_{B,0}(J) \to \mathcal{D}_{B,\gamma}(J)$ via universal property. To achieve this, we need to construct a (new) B-P.D. algebra structure on $\mathcal{D}_{B,0}(J)$ which is compatible with γ.
Indeed, denote by $\bar{\gamma}$ the extension of γ to B/J. The section $B/J \to B \to \mathcal{D}_{B,0}(J)$ of $\mathcal{D}_{B,0}(J) \to \mathcal{D}_{B,0}(J)/\bar{J} = B/J$ splits

$$\mathcal{D}_{B,0}(J) = B/J \oplus \bar{J}$$

Then $\bar{\gamma}$ extends to $\mathcal{D}_{B,0}(J)$ and this is a B-P.D. algebra structure compatible with γ.

Important remarks

(6) If $K \subset B$ is an ideal such that $K\mathcal{D}_{B,\gamma}(J) = 0$, then

$$\mathcal{D}_{B,\gamma}(J) = \mathcal{D}_{B/K,\gamma}(J/J \cap K)$$

□ Exercise of universal property.

For example, if $mB = 0$ for some integer m and J has $\leq q$ generators, then $J^{(m-1)q+1}\mathcal{D}_{B,\gamma}(J) = 0$ and hence

$$\mathcal{D}_{B,\gamma}(J) = \mathcal{D}_{B/J^{(m-1)q+1},\gamma}(J/J^{(m-1)q+1})$$

Geometrically, this means $\mathcal{D}_{B,\gamma}(J)$ only depends on infinitesimal neighborhood of $V(J)$ in $\mathrm{Spec}(B)$.

Important remarks

(7) (Base change) If $(A, I, \gamma) \to (A', I', \gamma')$ is a surjective P.D. morphism and $B' := A' \otimes_A B$, $J' := JB'$. Then the canonical map

$$A' \otimes_A \mathcal{D}_{B,\gamma}(J) \to \mathcal{D}_{B',\gamma'}(J') \text{ is an isomorphism}$$

$\square$ By the universal property of $\mathcal{D}_{B',\gamma'}(J')$, it suffices to see the image of $\bar{J}$ in $A' \otimes_A \mathcal{D}_{B,\gamma}(J)$ has a P.D. structure compatible with γ', or equivalently, the kernel of $\mathcal{D}_{B,\gamma}(J) \to A' \otimes_A \mathcal{D}_{B,\gamma}(J)$ meets $\bar{J}$ in a sub-P.D. ideal. If $A' = A/K$, then this kernel is $K\mathcal{D}_{B,\gamma}(J)$. To see $K\mathcal{D}_{B,\gamma}(J) \cap \bar{J} \subset \bar{J}$ is a sub-P.D. ideal, we use that $K \subset I$ is a sub-P.D. ideal and $[\ ]$ compatible with γ. Everything is then clear from Proposition-Definition (compatibility).

Change of algebra

Proposition (Change of algebra)

If B' is a B-algebra, then there is a natural map

$$\mathcal{D}_{B,\gamma}(J) \otimes_B B' \to \mathcal{D}_{B',\gamma}(JB')$$

which is an isomorphism if B' is flat over B.

Proof.

The above map comes from $\mathcal{D}_{B,\gamma}(J) \to \mathcal{D}_{B',\gamma}(JB')$, given by the universal property. If B' is flat over B, then $JB' = J \otimes_B B'$ and so $\Gamma_{B'}(JB') \cong \Gamma_B(J) \otimes_B B'$ is flat over $\Gamma_B(J)$. By definition

$$\mathscr{J}' = \mathscr{J}\Gamma_{B'}(JB') = \mathscr{J} \otimes \Gamma_B(J) \otimes B' \cong \mathscr{J} \otimes B'$$

$$\Downarrow$$

$$\mathcal{D}_{B',\gamma}(JB') = \Gamma_{B'}(JB')/\mathscr{J}' = \Gamma_B(J) \otimes B'/\mathscr{J} \otimes B' = \mathcal{D}_{B,\gamma}(J) \otimes_B B'$$

□

Change of algebra

Corollary (Extension of P.D. structure on flat case)

If B is a flat A-algebra, then γ extends to B.

Proof.

We have a map $\mathcal{D}_{A,\gamma}(I) \to A$ with a P.D. kernel. Tensored with B yields $\mathcal{D}_{A,\gamma}(I) \otimes_A B \to B$. By Proposition, this is $\mathcal{D}_{B,\gamma}(IB) \to B$. Let K be its kernel, then $K \cap \overline{IB} \subset \overline{IB}$ is a sub-P.D. ideal. Then apply Lemma (Criteria for extension of P.D. structure). □

Change of algebra

Corollary ($\mathcal{D}_{B,\gamma}(J)$ as $\mathbb{Q}$-algebra)

The map $B \to \mathcal{D}_{B,\gamma}(J)$ is an isomorphism mod $\mathbb{Z}$-torsion.

Proof.

Let $B' := B \otimes_{\mathbb{Z}} \mathbb{Q}$. Then B' is flat over B and $\mathcal{D}_{B',\gamma}(JB') = B'$ (since B' is now a $\mathbb{Q}$-algebra). By Proposition

$$B \otimes_{\mathbb{Z}} \mathbb{Q} = B' = \mathcal{D}_{B',\gamma}(JB') = \mathcal{D}_{B,\gamma}(J) \otimes_B B' = \mathcal{D}_{B,\gamma}(J) \otimes_{\mathbb{Z}} \mathbb{Q}$$

i.e. the map $B \to \mathcal{D}_{B,\gamma}(J)$ becomes an isomorphism if tensored $\mathbb{Q}$. $\square$

P.D. nilpotent

Definition (P.D. nilpotent)

Let (A, I, γ) be a P.D. ring. Then $I^{[n]}$ is the ideal generated by

$$\gamma_{i_1}(x_1)\cdots\gamma_{i_k}(x_k) \text{ for } \sum i_j \geq n \text{ and } x_j \in I$$

and I is called *P.D. nilpotent* if $I^{[n]} = 0$ for some $n \geq 1$.

Clearly $I^n \subset I^{[n]}$, then P.D. nilpotent $\Rightarrow$ nilpotent, but the converse is not true, e.g. $(2) \subset \mathbb{Z}/2^m\mathbb{Z}$ for some $m > 1$ (recall $x^n = n!\gamma_n(x)$).

Proposition

$I^{[n]} \subset I$ is a sub-P.D. ideal and $I^{[n]}I^{[m]} \subset I^{[n+m]}$.

Proof.

By Lemma (being a sub-P.D. ideal can be checked on a generating set), compute $\gamma_p(\gamma_{i_1}(x_1)\cdots\gamma_{i_k}(x_k)) \in I^{[n]}$ for any $p \geq 1$. □

Sheaf of P.D. rings

Definition

We can speak of a sheaf of P.D. rings on a topological space.

Definition (Sheaf of P.D. rings)

A sheaf of P.D. rings on a topological space X is a triple $(\mathscr{A}, \mathscr{I}, \gamma)$ where

- $\mathscr{A}$ is a sheaf of rings on X.
- $\mathscr{I} \subset \mathscr{A}$ is a sheaf of ideals on X.
- $\gamma = (\gamma_i : \mathscr{I} \to \mathscr{I})_{i \geq 0}$ is a collection of morphism of sheaves.

such that for any open $U \subset X$, the restriction

$$(\mathscr{A}(U), \mathscr{I}(U), \gamma(U)) \text{ is a P.D. ring}$$

Sheaf of P.D. rings

Basic operations

Let $f : X \to Y$ be a continuous map of topological spaces. The following constructions mean the obvious things.

1. (Push-forward) If $(\mathscr{A}, \mathscr{I}, \gamma)$ is a sheaf of P.D. rings on X, then $(f_*\mathscr{A}, f_*\mathscr{I}, f_*\gamma)$ is a sheaf of P.D. rings on Y.
2. (Pull-back) If $(\mathscr{B}, \mathscr{J}, \delta)$ is a sheaf of P.D. rings on Y, then $(f^*\mathscr{B}, f^*\mathscr{J}, f^*\delta)$ is a sheaf of P.D. rings on X.
3. and so on...

P.D. ringed space

Definition (P.D. ringed space & P.D. scheme)

- *A P.D. ringed space* is a pair $(X, (\mathscr{A}, \mathscr{I}, \gamma))$ where X is a topological space and $(\mathscr{A}, \mathscr{I}, \gamma)$ is a sheaf of P.D. rings on X.
- *A P.D. scheme* is a P.D. ringed space $(X, (\mathscr{A}, \mathscr{I}, \gamma))$ such that X is a scheme (with Zariski topology) and $\mathscr{A} = \mathcal{O}_X$.

Definition (Morphism of P.D. ringed spaces)

A morphism of P.D. ringed spaces $f : (X, (\mathscr{A}, \mathscr{I}, \gamma)) \to (Y, (\mathscr{B}, \mathscr{J}, \delta))$ consists of

1. a continuous map $f : X \to Y$ of topological spaces.
2. a morphism of sheaves of P.D. rings

$$f^{\#} : (\mathscr{B}, \mathscr{J}, \delta) \to (f_*\mathscr{A}, f_*\mathscr{I}, f_*\gamma)$$

In particular, it is a P.D. morphism when restricted to opens.

Building blocks: affine P.D. scheme

Let (A, I, γ) be a P.D. ring. Forgetting the P.D. structure, we get

an affine scheme $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ + a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathrm{Spec}(A)}$

defined by $\mathcal{O}_{\mathrm{Spec}(A)}(D(p)) = A_p$ and $\mathcal{I}(D(p)) = I_p$ for any $p \in A$.

Fact (P.D. structure on localization)

The localization pair (A_p, I_p) has a canonical P.D. structure γ given by

$$\gamma_n(x/p^i) := \gamma_n(x)/p^{in} \textit{ for any } x \in A \textit{ and } n \geq 0$$

Sheafification of this fact gives a P.D. structure on $\mathcal{I}$, hence yields a sheaf of P.D. rings $(\mathcal{O}_{\mathrm{Spec}(A)}, \mathcal{I}, \gamma)$ on $\mathrm{Spec}(A)$ which makes it into a P.D. ringed space, sometimes denoted by

$$\mathrm{Spec}(A, I, \gamma)$$

Again, there is a bijection

$$\mathrm{Hom}_{\mathsf{P.D.}}(\mathrm{Spec}(A, I, \gamma), \mathrm{Spec}(B, J, \delta)) = \mathrm{Hom}_{\mathsf{P.D.}}((B, J, \delta) \to (A, I, \gamma))$$

Sheafification of P.D. envelope

In the rest, we fix a base P.D. scheme $(S, (\mathcal{O}_S, \mathcal{I}, \gamma))$.

Proposition (Sheafification of P.D. envelope)

Given

- X*: an* S*-scheme.*
- $\mathcal{B}$*: a quasi-coherent sheaf of* $\mathcal{O}_X$*-algebras.*
- $\mathcal{J} \subset \mathcal{B}$*: a quasi-coherent sheaf of ideals.*

Then $\mathcal{D}_{\mathcal{B},\gamma}(\mathcal{J})$ *is a quasi-coherent sheaf of* $\mathcal{O}_X$*-algebras.*

In picture, we just replace

$$(A, I, \gamma) \xrightarrow{f} (B, J) \xrightarrow{\psi} (\mathcal{D}_{B,\gamma}(J), \bar{J}, [\ ])$$

$$\Downarrow$$

$$(\mathcal{O}_S, \mathcal{I}, \gamma) \xrightarrow{f} (\mathcal{B}, \mathcal{J}) \xrightarrow{\psi} (\mathcal{D}_{\mathcal{B},\gamma}(\mathcal{J}), \bar{\mathcal{J}}, [\ ])$$

P.D. neighbourhood

Definition

Let $i : X \hookrightarrow Y$ be a closed immersion of S-scheme defined by $\mathcal{I}_X \subset \mathcal{O}_Y$. In this case we write

$$\mathcal{D}_{X,\gamma}(Y) := \mathcal{D}_{\mathcal{O}_Y,\gamma}(\mathcal{I}_X) \text{ and } \mathcal{D}^n_{X,\gamma}(Y) := \mathcal{D}_{X,\gamma}(Y)/\bar{J}^{[n+1]}$$

Because of Proposition, they are quasi-coherent $\mathcal{O}_Y$-algebras.

Definition (n^{th}-order P.D. neighborhood of X in Y)

The schemes

$$i_n : \mathbf{D}^n_{X,\gamma}(Y) := \underline{\mathrm{Spec}}_Y(\mathcal{D}^n_{X,\gamma}(Y)) \to Y \text{ for } n \geq -1$$

are called *the n^{th}-order P.D. neighborhood of X in Y*.

Warning: This is not a subscheme of Y!

P.D. neighbourhood

Why the name?

In case that γ extends to $\mathcal{O}_Y/\mathcal{I}_X = \mathcal{O}_X$ (i.e. to X), then by Remark 4

$$\mathcal{D}_{X,\gamma}(Y)/\bar{J} = \mathcal{O}_X$$

and $i : X \hookrightarrow Y$ factors through a closed immersion $j : X \hookrightarrow \mathbf{D}_{X,\gamma}(Y)$ as

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & Y \\ & {\scriptstyle j}\searrow \qquad \nearrow{\scriptstyle i_n} & \\ & \mathbf{D}_{X,\gamma}(Y) & \end{array}$$

This somehow explains the name.

P.D. neighbourhood

Proposition (Local structure of $\mathcal{D}_{X,\gamma}(Y)$)

If $i : X \hookrightarrow Y$ is a (closed) immersion of smooth S-schemes and $m\mathcal{O}_Y = 0$. Then $\mathcal{D}_{X,\gamma}(Y)$ is locally isomorphic to a P.D. polynomial algebra over $\mathcal{O}_X$ generated by $d :=$ codim(X) elements.

In fact, we can compute it directly as

$$\begin{aligned}
\mathcal{D}_{X,\gamma}(Y) :&= \mathcal{D}_{\mathcal{O}_Y,\gamma}(\mathcal{I}_X) = \mathcal{D}_{\mathcal{O}_Y/\mathcal{I}_X^N,\gamma}(\mathcal{I}_X/\mathcal{I}_X^N) \text{ (Remark 6)} \\
&= \mathcal{D}_{\mathcal{O}_Y/\mathcal{I}_X^N,0}(\mathcal{I}_X/\mathcal{I}_X^N) \text{ (Remark 5)} \\
&= \mathcal{D}_{\mathcal{O}_X[t_1,\ldots,t_d]/\mathcal{I}^N,0}(\mathcal{I}/\mathcal{I}^N) \text{ } (X/S \text{ smooth}) \\
&= \mathcal{D}_{\mathcal{O}_X[t_1,\ldots,t_d],0}(\mathcal{I}) \text{ (explicit)} \\
&= \mathcal{O}_X\langle t_1,\ldots,t_d\rangle
\end{aligned}$$

where $N := (m-1)d + 1$ and $\mathcal{I} := (t_1,\ldots,t_d) \subset \mathcal{O}_X[t_1,\ldots,t_d]$.

Compatibility with inverse limit

Set-up

For many purposes it is convenient to work over a **formal base**: need to know the compatibilities of $\mathcal{D}_{\mathcal{O}_Y,\gamma}(\mathcal{J})$ with inverse limits. Given

- (A, I, γ): a Noetherian P.D. ring.
- $P \subset I$: a sub-P.D. ideal such that A is P-adically complete

By Corollary (Powers of P.D. ideals), $P^{n+1} \subset I$ is a sub-P.D. ideal and we have a natural P.D. morphism by Lemma (P.D. structure on a quotient ring)

$$(A, I, \gamma) \to (A_n, I_n, \gamma) := (A/P^{n+1}, IA/P^{n+1}, \gamma)$$

The P.D. structure on I_n induces a P.D. structure on

$$\hat{I} := \lim_{\leftarrow} I_n \subset \lim_{\leftarrow} A_n := \hat{A}$$

and finally yields a P.D. morphism $(A, I, \gamma) \to (\hat{A}, \hat{I}, \gamma) \to (A_n, I_n, \gamma)$. For any formal A-scheme Z, let $Z_n := Z \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$.

Compatibility with inverse limit

Statement

The following is just a global version of Remark 8.

Proposition (Compatibility of $\mathcal{D}_{\mathcal{O}_Y,\gamma}(\mathcal{J})$ with inverse limit)

Given

- Y*: a formal* A*-scheme with ideal of definition containing* $P\mathcal{O}_Y$ *such that* γ *extends to* Y*.*
- $\mathcal{J} \subset \mathcal{O}_Y$*: a sheaf of ideals.*

Then there are canonical isomorphisms

$$A_n \otimes_A \mathcal{D}_{\mathcal{O}_Y,\gamma}(\mathcal{J}) \xrightarrow{\sim} \mathcal{D}_{\mathcal{O}_{Y_n},\gamma}(\mathcal{J}\mathcal{O}_{Y_n})$$

$$\widehat{\mathcal{D}_{\mathcal{O}_Y,\gamma}(\mathcal{J})} \xrightarrow{\sim} \varprojlim \mathcal{D}_{\mathcal{O}_{Y_n},\gamma}(\mathcal{J}\mathcal{O}_{Y_n})$$

In particular, $\hat{\bar{\mathcal{J}}}$ *has a canonical P.D. structure compatible with* γ*.*

Compatibility with inverse limit

Geometric interpretation

The following is just a global version of Remark 7.

Corollary (The invariance of $\widehat{\mathbf{D}_{X,\gamma}(Y)}$ under formal completion)

Given

- Y*: a formal* A*-scheme with ideal of definition containing* $P\mathcal{O}_Y$ *such that* γ *extends to* Y*.*
- $X \subset Y$*: a formal subscheme.*

Assume that P *contains a non-zero integer. Then there is a canonical isomorphism*

$$\widehat{\mathbf{D}_{X,\gamma}(Y_{/X})} \to \widehat{\mathbf{D}_{X,\gamma}(Y)}$$

where $Y_{/X}$ *is the formal completion of* Y *along* X*.*

Compatibility with inverse limit

Local structure

Corollary (Local structure of $\widehat{\mathbf{D}_{X,\gamma}(Y)}$)

Given

- Y*: a formal* A*-scheme with ideal of definition containing* $P\mathcal{O}_Y$ *such that* γ *extends to* Y*.*
- $X \subset Y$*: a formal subscheme.*

Assume that Y/A *and* X/A_0 *are smooth. Then* $\widehat{\mathbf{D}_{X,\gamma}(Y)}$ *is locally isomorphic to the* P*-adic completion of a P.D. polynomial algebra with coefficient in a formally smooth* A*-algebra.*

In particular, if A *has no* $\mathbb{Z}$*-torsion, then* $\widehat{\mathbf{D}_{X,\gamma}(Y)}$ *has none.*

Thanks !