Lecture 11: THH and TC :Part II .

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Contents

Today, we shall at first view THH as a cyclotomic spectra and define the category CycSp and the functors $\mathrm{TC}, \mathrm{TC}^{-}$ and TP.

Recall: In the beginning of the seminar, we saw

$$\mathrm{HC}_{\bullet}(A) = H_{\bullet}(\mathrm{HochComplx}/(t_n - \mathrm{id})).$$

Also, we have a corresponding fiber sequence :

$$\mathrm{HC}[1] \to \mathrm{HC}^{-} \to \mathrm{HP}$$

At first, we start with Tate construction.

1 Tate Construction.

Tate Cohomology: For a finite group G and a G-module M, we have :

1. Group Coh : $R()^G$ (the right derived functor of invariants).
2. Group Hom: $L()_G$ (the left derived functor of orbits).

Tate noticed the existence of the map : $\mathrm{Nm} : M_G \to M^G$ defined by

$$\bar{a} \to \sum_{g \in G} ga$$

which allows us to patch these two cohomology theories.

The cohomology groups are

$$\cdots H_1(G, M), \ker(Nm), \operatorname{coker}(Nm), H^1(G, M), \cdots$$

One of the main useful of such a cohomology theory was to establish important results in Class Field theory, such as :

Theorem 1.0.1.
$$\hat{H}^{-2}(G, \mathbb{Z}) \xrightarrow[\cong]{\cup} \hat{H}^0(G, L^*) \cong K^*/N_{L/K}L^*$$

where L/K are local fields.

We want to generalize such notions to the category of G-spectra.

We have a trivial action functor $f^* : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}}^G$. This functor has left and right adjoints:

$$\mathrm{Mod}_{\mathbb{Z}} \underset{\underset{f_*}{\longleftarrow}}{\overset{\overset{f_!}{\longleftarrow}}{\xrightarrow{f^*}}} \mathrm{Mod}_{\mathbb{Z}}^G$$

where f_* is functor of fixed points and $f_!$ is the functor of orbits.

Definition 1.0.1. For a G- spectra X, i.e $X : \mathrm{BG} = N(G) \to \mathrm{Sp}$, we define:

$$(X)^{hG} = f_*(X) := \lim_{BG} X$$

and

$$(X)_{hG} = f_!(X) := \mathrm{colim}_{BG} X$$

We have the following theorem.

Theorem 1.0.2. *We have a natural transformation* $\mathrm{Nm} : ()_{hG} \to ()^{hG}$.

Definition 1.0.2. $()^{tG} := \mathrm{cofib}(\mathrm{Nm} : ()_{hG} \to ()^{hG})$. $()^{tG}$ is the called the Tate spectra.

Remark 1.0.1. Applying the functor $()^{tG}$ to HM for a G-module M, we get the homotopy groups of this spectra to be the tate cohomology groups.

With this functors defined, we can define the following notions:

Definition 1.0.3. For A a ring, we define :

1. $\mathrm{TC}^-(A) := \mathrm{THH}(A)^{h\mathbb{T}}$.
2. $\mathrm{TP}(A) := \mathrm{THH}(A)^{t\mathbb{T}}$.

Turns out that to define, TC, we need one more tool ,i.e the Tate diagonal.

1.1 Tate diagonal

In Sp, it is not obvious to have a diagonal functor :

$$X \to X \otimes X$$

But for all prims p, we have a functor called the **Tate diagonal**, denoted by

$$\Delta_p : X \to (X^{\otimes p})^{tC_p}$$

for each prime p.

Remark 1.1.1. Fact: the functor $T_p : \mathrm{Sp} \to \mathrm{Sp}$ is exact where

$$T_p(X) = (X^{\otimes p})^{tC_p}$$

In fact, we have

$$T_p(X \oplus Y) = T_p(X) \oplus T_p(Y) (\oplus = \text{wedge}).$$

We have the following lemma in the category of Sp,

Lemma 1.1.1. *Let $F : \mathrm{Sp} \to \mathrm{Sp}$ be an exact functor, evaluation at $\mathbb{S}$ gives us the following equivalence:*

$$\mathrm{Map}_{\mathrm{Fun}^{ex}(\mathrm{Sp},\mathrm{Sp})}(\mathrm{id}_{Sp}, F) \xrightarrow{\cong} \mathrm{Hom}_{Sp}(\mathbb{S}, F(\mathbb{S})).$$

Definition 1.1.1. The natural transformation $\mathrm{id}_{Sp} \to \mathrm{Tp}$ corresponding to the map $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p} = T_p(\mathbb{S})$ is called the Tate diagonal.

Note: The map only exists in higher algebra.

Remark 1.1.2. Given $A \in \mathrm{Alg}_{E_1}(\mathrm{Sp})$, we have the following diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\Delta_p} & (A^{\otimes p})^{tC_p} & \longrightarrow & A^{tC_p} \\ \downarrow & & & & \downarrow \\ \mathrm{THH}(A) & & \xrightarrow{\varphi_p} & & \mathrm{THH}(A)^{tC_p} \end{array}$$

This gives us the fact that $\mathrm{THH}(A)$ is a $\mathbb{T}/C_p \cong \mathbb{T}$-spectrum.

Definition 1.1.2. A cyclotomic spectrum is a $\mathbb{T}$-spectrum X equipped with the maps $X \xrightarrow{f_p} X^{tC_p}$ for each prime p which are $\mathbb{T}$-equivariant.

Example 1.1.1. For $A \in \mathrm{Alg}_{E_1}(\mathrm{Sp})$ with φ_p as above gives $\mathrm{THH}(A)$ as a cyclotomic spectra.

This gives us the definition of TC.

Definition 1.1.3. For X a cyclotomic spectra, we define :

$$\mathrm{TC}(X) := \mathrm{Eq}(X^{h\mathbb{T}} \underset{\prod f_p}{\overset{\mathrm{can}}{\rightrightarrows}} (X^{tC_p})^{h\mathbb{T}})$$

2 Defining CycSp.

1. $\mathcal{C} = \mathrm{Mod}_{\mathbb{Z}}, \ \mathrm{Hom}(1, X) = X.$
2. $\mathcal{C} = \mathrm{Mod}^G, \ \mathrm{Hom}_{\mathcal{C}}(1, X) = X^G.$
3. $\mathcal{C} = \mathrm{Sp}, \ \mathrm{Hom}_{\mathcal{C}}(\mathbb{S}, X) = \Omega^\infty X.$

We want to define the category of CycSp, in such a way such that TC is the mapping spectrum motivated from the previous example.

Definition 2.0.1. CycSp is the lax equalizer of $\mathrm{Sp}^{\mathbb{T}} \underset{\prod_p (.)^{tC_p}}{\overset{\mathrm{id}}{\rightrightarrows}} \prod_p \mathrm{Sp}^{\mathbb{T}}$ In other words, we define the category CycSp can be defined as the pullback diagram:

$$\begin{array}{ccc} \mathrm{CycSp} & \longrightarrow & (\prod \mathrm{Sp}^{\mathbb{T}})^{\Delta 1} \\ \downarrow & & \downarrow \\ \mathrm{Sp}^{\mathbb{T}} & \xrightarrow{\prod_p \phi_p} & \prod_p \mathrm{Sp}^{\mathbb{T}} \times \prod_p \mathrm{Sp}^{\mathbb{T}} \end{array}$$

where $\phi_p = (\mathrm{id}, \prod_p (.)^{tC_p})$.

In this way, we can define $\mathrm{TC}(X) = \mathrm{Map}_{\mathrm{CycSp}}(\mathbb{S}^{triv}, X)$.

Definition 2.0.2. For a ring A, we define $\mathrm{TC}(A) := \mathrm{TC}(\mathrm{THH}(A))$.

Two main important remarks about such theory:

1. We have a following diagram:

$$\begin{array}{ccc} K & \xrightarrow{\mathrm{Dtr}} & \mathrm{THH} \\ & \searrow^{\mathrm{Cyct}} & \uparrow \\ & & \mathrm{TC} \end{array}$$

   where Dtr is called the **Dennis trace map** and Cyct is called the **Cyclotomic trace map**.

2. The cyclotomic trace map has a following important consequence given by the following theorem

   Theorem 2.0.1 (Dundas-Goodwille-McCarthy)**.** *$A \twoheadrightarrow B$ a surjective maps of rings with kernel a nilpotent ideal., then the following square is cartesian:*

$$\begin{array}{ccc} K(A) & \longrightarrow & \mathrm{TC}(A) \\ \downarrow & & \downarrow \\ K(B) & \longrightarrow & \mathrm{TC}(B) \end{array}$$