Babyseminar WS 2020/21 : Crystalline and Prismatic Cohomology.

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Contents

1 Introduction.

The goal of the seminar is to understand two cohomology theories : Crystalline and Prismatic Cohomology. These two cohomology theories are an important tool in the field of arithmetic geometry. The seminar is divided into two parts as the name suggests:

1. Lectures $1-7$: **Crystalline Cohomology**.
2. Lectures $8-12$: **Prismatic Cohomology**.

Each lecture is of 90 mins. There are some lectures where the speaker may need 15-30 minutes extra for completing the talk.

Let us give a brief introduction to both of these theories.

1.1 Crystalline Cohomology.

"Un cristal posséde deux propriétés caractéristíques:la rigidité, et la faculté de croitre, dans un voisinage approprié. Il y a des cristaux de toute espéce de substance: des cristaux de soude, de souffre, de modules, d'anneaux, de schémas relatifs etc"
- Grothendieck, in a letter to Tate, 1966.

Crystalline Cohomology was invented by Grothendieck in 1966, (as you can see above) in order to find a "good" p-adic cohomology theory. It was worked out by Pierre Berthelot in his thesis. To give a motivation to this, let us briefly describe some properties of l-adic cohomology.
The l-adic cohomology developed by Grothendieck and Artin is a Weil Cohomology theory, in the sense that it is a contravriant functor H^* on the category of smooth projective varieties over $k = \mathbf{F}_q$ of characteristic p ($l \neq p$) to the category of graded algebras over a field $\mathbf{Q}_l$ of characteristic 0 which satisfies a bunch of properties. Some of the important properties are :

1. Poincare duality.
2. Trace map.
3. Cycle map.
4. Weak and Hard Lefschetz theorem.

Morever, for X a smooth projective scheme over $k = \mathbf{F}_p$ which admits a smooth proper lift $\mathcal{X}$ over $\mathbf{Z}_p$, we have

$$H^*(X_{\bar{k},et}, \mathbf{Z}_l) = H^*_{\text{sing}}(\mathcal{X}^{\text{an}}, \mathbf{Z}) \otimes \mathbf{Z}_l$$

where we fixed an embedding $\mathbf{Z}_p \hookrightarrow \mathbf{C}$. This is a consequence of Artin's comparison theorem in SGA IV involving finite coefficents and passing to inverse limits. An important fact here to note is that the comparison map does not kill the l-torsion information of the singular cohomology .

But what about the case $l = p$? Turns out that almost none of the above properties hold for p-adic étale cohomology of varieties in characteristic p. This motivates the development of crystalline cohomology.

Firstly, Grothendieck introduced the notion of "infinitesimal sites". In short, for a variety X over $\mathbf{C}$, he defines a site $X/\mathbf{C}_{\text{inf}}$ which encodes the data of infinitesimal liftings. Objects of $X/\mathbf{C}_{\text{inf}}$ are pairs (U, T) where $U \subset X$ is an open subset and $U \to T$ is a closed nilpotent immersion, namely they have the same underlying topological space such that $\mathcal{O}_T \to \mathcal{O}_U$ is quotient by a nilpotent ideal. The topology is given by Zariski covering.

A sheaf $\mathscr{F}$ on $X/\mathbf{C}_{\text{inf}}$ is given by a collection of sheaves $\mathscr{F}_T$ on each (U, T) such that for $f : (U, T) \to (U', T')$ it induces a morphism $f^*\mathscr{F}_{T'} \to \mathscr{F}_T$. If all such induced morphisms are isomorphisms, $\mathscr{F}$ is said to be a *crystal*. Thus coming to the quotation above, a crystal is an object which is "rigid" (i.e all such induced maps are isomorphisms) and it "grows" (i.e it can be extended along infinitesimal thickenings).

Grothendieck, in *Crystals and De Rham Cohomology of Schemes*, proved that

$$H^*(X/\mathbf{C}_{\inf}, \mathcal{O}_{X/\mathbf{C}}) \cong H^*(X, \Omega^\bullet_{X/\mathbf{C}}).$$

It is natural to apply such topology in our equicharacteristic situation, but it turns out that Poincare lemma fails in our setting. The obstruction to such proof yields an idea of how to resolve the issue. Thus, Grothendieck defines the notion of *divided power structures* and notion of *crystalline site*, where the cohomology of the site is called *crystalline cohomology.*

The seminar shall introduce the notion of divided power structures, the crystalline site and crystalline cohomology. We can define divided power ideals as follows:

Definition 1.1 ([1], Definition 3.1)**.** Let A be a commutative ring, $I \subset A$ be an ideal. By *divided powers on* I, we mean a collection of maps $\gamma_i : I \to A$ for all integers $i \geq 0$ such that $n!\gamma_n(x) = x^n$ for any $n \geq 0$ and for all $x \in I$.

A very important example is the following:

Example 1.1. Let k be a field of characteristic p. Let $W = W(k)$ be the ring of Witt vectors. Then the pair $(p) \subset W$ has a divided power structure given by $\gamma_n(x) = x^n/n!$ for $x \in (p)$.

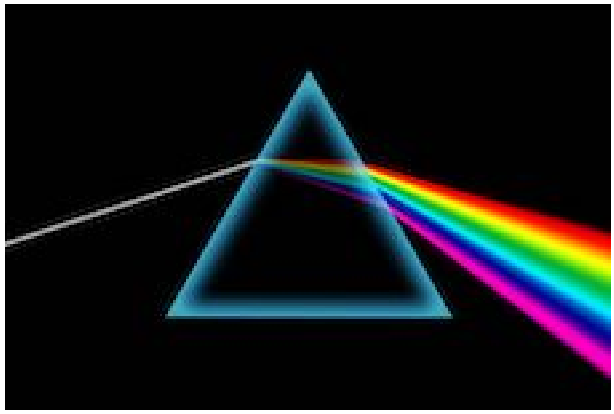
Such notions can be upgraded on the level of schemes and formal schemes. This shall enable us to define the crystalline site $(X/S)_{\text{cris}}$ where S is a P.D scheme and locally nilpotent. Categorical formalism allows us to define crystalline cohomology. We shall introduce the definition of *crystals* in the setting of Crystalline site. The main goal of the first part of our seminar is to prove the comparsion theorem between crystalline and de Rham cohomology.

Theorem 1.1 ([2], Corollary 3.8)**.** *Let k be a perfect field of characteristic p. Let $W = W(k)$ be the ring of Witt vectors. Let X be a smooth variety over k. Suppose X admits a smooth proper lifting Z over W, then*

$$H^i_{\text{cris}}(X/W) \cong H^i_{\text{dR}}(Z/W).$$

1.2 Prismatic Cohomology.

The second part of the seminar is devoted to the study of prismatic cohomology, defined by Bhatt and Scholze in [5]. To say it in their words: prismatic cohomology is a "unified construction of various p-adic cohomology theories, including étale, de Rham and crystalline cohomology...". Just like one obtains light of different colours when sending white light through a prism, one obtains the already existing p-adic cohomology theories from the prismatic one, whence the name, see Terence Tao's blogpost [9].



Given a ring A together with a ring endomorphism ϕ on it, which lifts the Frobenius modulo p, we obtain a set theoretic map $\delta\colon A \to A$ such that $\phi(a) = a^p + p\delta(a)$ for all $a \in A$. The fact that ϕ is a ring morphism is encoded in certain identities that δ satisfies. If A is p-torsionfree, then giving ϕ is equivalent to giving the map δ; if however A has p-torsion, then at least any map of sets $\delta\colon A \to A$ satisfying the aforementioned identities yields a Frobenius lift ϕ by the above formula. Pairs (A, δ) are called δ-rings.

Example 1.2. Let k be a perfect field of characteristic $p > 0$. Then the Frobenius on k induces a Frobenius lift on $W(k)$. The kernel of the canonical map $W(k) \twoheadrightarrow k$ is principal, generated by p. The pair $(W(k), (p))$ consisting of the p-adically complete δ-ring $W(k)$ and the ideal (p) is an instance of a *crystalline prism*.

Definition 1.2 ([5], Definition 3.2)**.** A *prism* is a pair $((A, \delta), I)$ consisting of a δ-ring (A, δ) (with associated Frobenius ϕ) and an ideal $I \subset A$, which defines a Cartier divisor in $\mathrm{Spec}(A)$, such that

(i) A is (derived) (p, I)-adically complete,

(ii) $p \in I + \phi(I)A$, i.e. $V(I)$ and $(\phi^*)^{-1}(V(I))$ meet only in characteristic p.

Example 1.3. The canonical map $W(\mathcal{O}^\flat_{\mathbf{C}_p}) \twoheadrightarrow \mathcal{O}_{\mathbf{C}_p}$ from p-adic Hodge theory defines a prism. Indeed, the kernel is generated by the element $\xi = p - [p^\flat]$, where $p^\flat \in \mathcal{O}^\flat_{\mathbf{C}_p} := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbf{C}_p}$ is a compatible system of p-power roots of p. Since $\phi(\xi) - \xi^p \in p(1 + pW(\mathcal{O}^\flat_{\mathbf{C}_p}))$, we see that $\delta(\xi)$ is a unit in $W(\mathcal{O}^\flat_{\mathbf{C}_p})$ and thus condition (ii) is satisfied.

The ring $\mathcal{O}_{\mathbf{C}_p}$ is an example of a perfectoid ring, and the Frobenius on $W(\mathcal{O}^\flat_{\mathbf{C}_p})$ is an isomorphism, so that the $(W(\mathcal{O}_{\mathbf{C}^\flat_p}), (\xi))$ is an instance of a *perfect* prism.

We will define perfectoid rings in general and then see the following result:

Theorem 1.2 ([5], Theorem 3.10)**.** *Mapping a perfect prism (A, I) to A/I induces an equivalence of categories*

$$(\text{Perfect prisms}) \simeq (\text{Perfectoid rings})\,.$$

The inverse is given by sending a perfectoid ring S to the prism $(W(S^\flat), \ker(\theta))$, where $\theta\colon W(S^\flat) \to S$ is Fontaine's canonical map, which will be recalled in one of the talks.

Going back to the crystalline setting in Example 1.2, given an algebraic variety over k admitting a smooth lift to $W(k)$, we have the attached crystalline complex $R\Gamma_{\mathrm{crys}}(X/W(k))$, computed as the sheaf cohomology of the structure sheaf on the crystalline site $(X/W(k))_{\mathrm{crys}}$. In a similar fashion, restricting to the affine situation, given a prism (A, I) and a smooth p-complete A/I-algebra R, there is the prismatic site $(R/A)_{\Delta}$ along with a structure sheaf $\mathcal{O}_{\Delta}$. Our main goal is to explain the comparison result:

Theorem 1.3 ([5], Theorem 5.2)**.** *Let $(A, (p))$ be a crystalline prism, let $I \subset A$ be a pd-ideal containing p. Given a smooth A/I-algebra R, there is a canonical isomorphism*

$$\Delta_{R^{(1)}/A} \simeq R\Gamma_{\mathrm{crys}}(R/A),$$

where $R^{(1)} = R \otimes_{A/I,\phi} A/p$ and $\Delta_{R^{(1)}/A} = R\Gamma((R^{(1)}/A)_{\Delta}, \mathcal{O}_{\Delta})$.

The key ingredients for the proof of this theorem are the realization of divided powers via δ-structures as well as the prismatic envelope of a δ-pair, paired with the fact that cohomology on a chaotic site can be computed via a Čech-Alexander complex.

We will also introduce the Hodge-Tate comparison map; and, if time permits, we will also see how to apply the crystalline comparison to show that the Hodge-Tate comparison map is an isomorphism in characteristic p (we will not see a proof in the characteristic 0 case, though):

Corollary 1.1 ([5], Corollary 5.4)**.** Given a smooth A/p-algebra S, the Hodge-Tate comparison map

$$\Omega^*_{S/(A/p)} \cong H^*(\overline{\Delta}_{S/A})$$

is an isomorphism, where $\overline{\Delta}_{S/A} = \Delta_{S/A} \otimes^L_A A/I$.

Prismatic cohomology seems to be much more powerful than we can appreciate during this seminar, but we hope that this program gives a helpful mild introduction to it, while relating it to what we will have learned about crystalline cohomology in the first part of the seminar.

Convention: Whenever there is no precise reference given in the detailed description of a talk, we implicitly refer to the main reference for that talk.

2 Outline of the talks.

PART I :CRYSTALLINE COHOMOLOGY.

2.1 Divided Powers : I.

Overview: This talk introduces the building blocks for defining crystalline cohomology, i.e. "divided powers". We shall learn basic terminologies related to divided powers on the level of rings. The talks shall end with defining "extension of P.D structures along Algebras" and notion of compatibility of P.D structures.

Main reference: [1, Chapter 3, 3.1–3.18 , Appendix A].

In detail: Introduce the basic definiton with examples (3.1-3.2). Define the notion of "sub P.D ideals" (3.4) and prove lemmas following it. Define "sub P.D algebras"(3.7). Introduce Roby's divided power envelope of a module $\Gamma_A(M)$ from Appendix A and state Theorem (3.9). Prove and state the propositions(3.10-3.12). Introduce the notion of "set of P.D generators of I" (3.13), extension of algebras (3.14) and notion of compatibility(3.17). Also state and sketch the proof of propositions (3.15-3.16).

2.2 Divided Powers : II.

Overview : This lecture continues the notion of divided powers on level of the rings and extend it on level of formal schemes. The talk starts with defining analogue of formal completion in divided power setup. We also introduce notion of ideals being "P.D nilpotent". We then move to level of schemes and define divided power envelope of X in Y , where $X \hookrightarrow Y$ is a closed immersion of schemes. We study the structure of these envelopes in the usual setting as well as in the setting of formal schemes.

Main reference: [1, Chapter 3, 3.18–3.35].

In detail: Introduce the algebra $\mathcal{D}_{B,\gamma}(J)$,state Theorem 3.19 and sketch the proof of it. State remark 3.20. State the definitions, propositions and corollaries (3.21-3.29). Give a sketch of the proofs of the propositions and corollaries mentioned above. Explain the notion of P.D sheaf of rings and $\text{Spec}(A, I, \gamma)$. Explain the setup of closed immersion of schemes and sketch the proof of proposition (3.30-3.32). Do the same in the setting of formal schemes (3.33-3.35).

2.3 The Crystalline Topos : I.

Overview: This lecture introduces the notion of Crystalline site and its basic properties. The main statement that we will prove is to associate a morphism of topoi corresponding to a PD morphism $S' \to S$. We shall define global section functors and cohomology functors in the crystalline setting. The talk shall end with stating the rigidity property of crystalline cohomology.

Main reference: [1, Chapter 5, 5.1–5.17] and [3, Lecture 5, Section 4] .

In detail: Define the crystalline site and explain the notion of sheaves and presheaves on the site. Prove proposition 5.1. State the examples 5.2. Recall the definition of morphism of topoi (5.4). Define the pullback functor (5.6). Prove proposition 5.7 and sketch the proof of proposition 5.8. Sketch the proof of proposition 5.9 via stating the statements 5.10 -5.13.
Define the global section functor (5.15) and discuss it in the crystalline setting. Define the cohomology functors H^i. Give a sketch of proof of rigidity along certain PD thickenings (5.17) using proposition 5.16.

2.4 The Crystalline Topos : II.

Overview : This talk continues the discussion on crystalline topos. More precisely, we shall understand about the usual morphism of topoi $(X/S)_{\mathrm{cris}} \to X_{\mathrm{Zar}}$. We shall understand the localization functor $(X/S)_{\mathrm{cris}}|_{\tilde{Z}} \to (X/S)_{\mathrm{cris}}$ where $\tilde{Z} = \mathrm{Hom}(-, Z)$ the usual representable sheaf and its consequences on the cohomology. .

Main reference: [1, Chapter 5, 5.18–5.29].

In detail: State Proposition 5.18 which describes the morphism of topoi from Crystalline to Zariski. Define the section functor(5.19) and its consequence in cohomology (5.20). Briefly sketch the proof of propositions 5.22-5.24 which explains canonical isomorphism in cohomology of localization in any topos. Explain the description of localization in crystalline setting and sketch the proof of propositions and corollaries (5.25-5.27).

2.5 Crystals I.

Overview: This lecture introduces the notions of crystals and introduces the notion of sheaf of differentials in the divided power setting. In particular, we shall see that the sheaf of differentials are **not** crystals but they are close to being a crystal. We shall also come across the notion of divided De Rham Complex in the divided power setting of rings.

Main reference: [8, Chapter 59, Section 6, 11 and 12] and [1, Chapter 6].

In detail: Define crystals ([8, Chapter 59, Section 11.1]) and prove lemma 11.2. State the definition of crystal in quasi-coherent $\mathcal{O}_X$-modules (Definition 11.3). State proposition 6.2 from [1, Chapter 6].
Introduce the notion of divided power derivations from [8, Chapter 59, Section 6.1]. Sketch the proof of lemmas 6.2-6.6 from [8, Chapter 59]. Define the divided power de Rham complex ([8, Chapter 59, Remark 6.7]).
Move to the section of sheaf of differentials from [8, Chapter 59, Section 12]. Introduce the notion of S-derivations [8, Chapter 59, Definition 12.1] and define $\Omega_{X/S}$ in the crystalline setting. State lemmas 12.2 -12.5 in loc. cit. Sketch the proof of Lemmas 12.6 in loc. cit to prove that $\Omega_{X/S}$ is not a crystal.

2.6 Crystals II.

Overview : This lecture continues the discussion of quasi-coherent crystals from the previous lecture. Particularly, the aim is to understand the category of quasi-coherent crystals on $(X/S)_{\text{cris}}$ in terms of " quasi-nilpotent integrable connections" where X and S are affine. This shall help us to prove the comparison theorem between Crystalline and De Rham Cohomology.
The lecture shall introduce general notion of Cohomology of Categories and introduce the Čech-Alexander Complex associated to a general topos. We shall use it to prove that Crystalline Cohomology can be computed by such a complex. Morever, the notions shall be also used in Prismatic Cohomology in second part of our seminar.

Main reference: [8, Chapter 59, Sections , 14, 15, 17 and 18], [3, Chapter 5, Section 4] and [2, Section 2].

In detail: Define the De Rham Complex in the small crystalline site ([8, Chapter 15, Section 14]). Introduce the notions of connections ([8, Chapter 15, Section 15]) and prove Lemma 15.1 in loc. cit. Move to [8, Chapter 59, Section 17] and introduce the notations D, Ω_D and $D(n)$. Introduce the conditions $(1), (2), (3)$ and (4) for [8, Chapter 59, Lemma 17.3]. State and sketch the proof of lemma 17.3 in loc. cit. State and sketch the proof of Proposition 17.4 in loc. cit.
Describe the notion of Cohomology of Categories from [3, Chapter 5, Section 4] and define the Čech Alexander Complex (the complex defined in [3, Chapter 5, Lemma 4.3]). State [8, Chapter 59, Lemma 18.2] in the crystalline site. Use the above lemmas to deduce the definition of Čech-Alexander Complex given in [2, Notation 2.1] and also deduce [2, Lemma 2.4] as a consequence of lemmas mentioned above.

2.7 The Comparison theorem.

Overview : In this lecture, we shall give a sketch of the prove of the comparison theorem, which is : Let X be a smooth variety over $\mathbb{Z}/p$, then the crystalline cohomology of X is canonically identified as the de Rham cohomology of a lift of X over $\mathbb{Z}_p$ provided it exists. We give a sketch of the proof on affine case. The talk shall end with briefly discussing the theorem in the global case.

Main Reference: The main reference is [2].

In detail: Recall the Čech-Alexander Complex as in [2, Notation 2.1] as done in the previous lecture. Move to the proof on the affine case and state the lemmas and explain the proof of theorem 2.12 (2.12-2.17). Also recall the notions of simplicial modules in the context if time permits (2.16). Move to section 3 and state theorems 3.2 and 3.6. State remark 3.7 and prove corollary 3.8 using theorem 3.6.

PART II : PRISMATIC COHOMOLOGY.

2.8 Basics on δ-rings.

Overview: The aim of this talk is to introduce δ-rings and establish some basic results for them, namely that the category of δ-rings has all limits and colimits, there are free objects, there is a good notion of quotient δ-rings, and the δ-structure extends to localizations and completions. The talk ends with the discussion of distinguished elements.

Main reference: [5, Chapter 2].

In detail: Give the definition of a δ-ring (Definition 2.1) and provide some examples, see for instance [3, Lecture 2], explain Remarks 2.2 and 2.4, Example 2.6. Discuss limits, colimits of δ-rings and free δ-rings (Remark 2.7, Lemma 2.11) and finally discuss quotients of δ-rings, see 2.8, 2.9 and 2.10. Explain how to extend δ-structure to localization and completions (Lemma 2.15 and 2.17).

Introduce distinguished elements (Definition 2.19) and give the examples 2.20 and 2.21. Prove Lemma 2.25.

2.9 Perfect δ-rings and prisms.

Overview: The first part of the talk is concerned with those δ-rings whose associated Frobenius is an isomorphism, i.e. perfect δ-rings. It will cover the equivalence of categories between p-complete perfect δ-rings and perfect $\mathbf{F}_p$-algebras, via $(A, \delta) \mapsto A/p$. We will also see that a distinguished element in a perfect δ-ring (and more generally in a p-torsionfree, p-adically separated δ-ring, whose reduction mod p is reduced) is a nonzerodivisor if and only if the quotient by that element has bounded p^∞-torsion.

The second part of the talk introduces prisms together with examples and establishes some first properties of them, such as the rigidity of prisms.

Main reference: [5, Chapter 2 and 3].

In detail: Cover section 2.4 on perfect δ-rings, in particular prove the equivalence of categories Corollary 2.31. You can skip Remark 2.30 and Lemma 2.32; prove Lemma 2.34.

Define prisms (Definition 3.2)[1], prove Lemma 3.1 and give examples (e.g. Example 3.3, 3.4 and [3, Lecture 5, p.1]). State the rigidity of prisms, Lemma 3.5, and state Lemma 3.6 and finish by proving Lemma 3.8.

[1] We do not plan to discuss derived completeness. Since we will only deal with bounded prisms and these are classically complete, this should be good enough for our purpose.

2.10 Perfectoid rings.

Overview: This talk covers basics on perfectoid rings with examples. In particular, we will see equivalent ways of defining a perfectoid ring. Along the way, we recall Fontaine's infinitesimal ring and the canonical map from p-adic Hodge theory coming from the adjunction between the ring of Witt vectors and tilting. Finally, we will establish an equivalence of categories between perfectoid rings and perfect prisms, given by mapping a perfect prims (A, I) to A/I, using results from the previous talk.

Main reference: [4, Chapter 3]

In detail: Following §3.1 define the tilt of a p-adically complete ring and introduce Fontaine's ring; state Lemma 3.2 (i). Mention that the functor of Witt vectors $W(-)$, as a functor from perfect $\mathbf{F}_p$-algebras to p-adically complete $\mathbf{Z}_p$-algebras, is left adjoint to the tilting functor. Define Fontaine's (surjective) map to be the counit of this adjunction, and write it down explicitly, see for example Lemma 3.3. Write down what its kernel is in the example of $\mathcal{O}_{\mathbf{C}_p}$ from p-adic Hodge theory, see [7, Proposition 5.12].

Define perfectoid rings, Definition 3.5, sketch a proof of Lemma 3.10, and discuss some examples of perfectoid rings, e.g $\mathcal{O}_{\mathbf{F}_p}$, Example 3.6 or others... Explain the equivalence of categories between perfect prisms and perfectoid rings, [5, Theorem 3.10].

2.11 Prismatic site and Hodge-Tate comparison map.

Overview: In this talk we will define the prismatic site, as well as the prismatic complex computing prismatic cohomology and the Hodge-Tate complex. The latter will be related to de Rham cohomology via the Hodge-Tate comparison map.

Main reference: [3, Lecture 5]

In detail: Define the prismatic site of R relative to A, $(R/A)_{\Delta}$, where (A, I) is a fixed base prism and R is a p-completely smooth[2] A/I-algebra, and define the structure sheaves $\mathcal{O}_{\Delta}$ and $\overline{\mathcal{O}}_{\Delta}$, Definition 2.1. Give the examples 2.6, 2.7. Then define the prismatic complex $\Delta_{R/A}$ and the Hodge-Tate complex $\overline{\Delta}_{R/A}$, Definition 2.10, and mention Example 2.11.

Justify the name "Hodge-Tate" complex by introducing the Hodge-Tate comparison map: Restrict yourself to the crystalline setting (in particular there is no need for derived p-completions of the modules of differentials). Follow §3.1 and §3.2, and finish by stating Theorem 3.8.

[2] See [3, Lecture 5, footnote 1] for the definition of p-complete smoothness; in the crystalline case this just means that R is smooth as an algebra over A/p-algebra.

2.12 The crystalline comparison isomorphism.

Overview: In this final talk we will see how crystalline cohomology can be recovered from prismatic cohomology. The key ingredient will be to realize divided powers via δ-structures, defining the prismatic envelope of a δ-pair and then comparing the crystalline and prismatic site using Čech-Alexander complexes for both sides.

If time permits, we will also sketch a proof of the Hodge-Tate comparison map being an isomorphism in the characteristic p setting, using the crystalline comparison theorem.

Main reference: [3, Lectures 5 and 6] and [5, Chapter 5]

In detail: Following [3, Lecture 5], define the prismatic envelope of a δ-pair, Lemma 5.1, give a sketch of Corollary 5.2, and discuss the Čech-Alexander complex for prismatic cohomology (Construction 5.3).

Recall divided power envelopes, see e.g. Construction 1.1 and Lemma 1.2 in [3, Lecture 6]. Explain how to realize divided powers via δ-structures, i.e. prove [5, Corollary 2.38] by sketching a proof of Lemma 2.37 in loc. cit. Use it to prove the crystalline comparison theorem [5, p. 5.2] (Emerton's notes on this, [6, Lecture 19], might be useful; see also [3, Lecture 6, Theorem 3.2]). Deduce a Frobenius descent result for the crystalline complex, [5, Remark 5.3].

Optional[3]*:* If enough participants are willing to stay 30 minutes longer, we can sketch a proof of the Hodge-Tate comparison theorem in characteristic p. Namely, explain Cartier's isomorphism, Construction 1.9 in [3, Lecture 6]; and use it to sketch a proof of [5, Corollary 5.4], see also [3, §4, Lecture 6].

References

[1] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978, pp. vi+243. ISBN: 0-691-08218-9.

[2] Aise Johan De Jong Bhargav Bhatt. *Crystalline Cohomology and De Rham Cohomology.* https://arxiv.org/abs/1110.5001. 2011.

[3] Bhargav Bhatt. *Prismatic cohomology (Eilenberg Lectures at Columbia University).* http://www-personal.umich.edu/~bhattb/teaching/prismatic-columbia/. 2018.

[4] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Integral p-adic Hodge theory". In: *arXiv e-prints*, arXiv:1602.03148 (Feb. 2016), arXiv:1602.03148. arXiv: 1602.03148 [math.AG].

[5] Bhargav Bhatt and Peter Scholze. *Prisms and Prismatic Cohomology.* 2019. eprint: arXiv:1905.08229.

[3] We could also split up the talk into two parts, so that someone else can explain this optional part.

[6] Matthew Emerton. *Lecture Course on Prismatic Cohomology*. http://math.uchicago.edu/~emerton/prismatic/prismatic.html. Spring 2020.

[7] Jean-Marc Fontaine and Yi Ouyang. *The Theory of p-adic Galois representations*. https://www.imo.universite-paris-saclay.fr/~fontaine/galoisrep.pdf.

[8] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2020.

[9] Terence Tao. *Blogpost on Prismatic Cohomology*. https://terrytao.wordpress.com/2019/03/19/prismatic-cohomology/. March, 2019.

That's all folks !!